Chapter 1 First Order Differential Equations

Jeaheang(Jay) Bang

Rutgers University

j.bang@rutgers.edu

Monday, July 10, 2017

Overview

1. Course description
2. Chapt. 1 First-Order Differential Equations
3. 1.1 Modeling via Differential Equations
   - What is a Model?
   - Unlimited Population Growth
   - Limited Resources and the Logistic Population Model
   - Predator-Prey Systems
   - The Analytic, Qualitative, and Numerical Approaches
   - Homework

About me

- Name: Jeaheang(Jay) Bang
- Office: Hill Center 603
- Office Hour: Monday, Wednesday 2-3pm at Hill 603
- Research: PDE(Partial Differential Equations)

Course Description

- Exam: one midterm, one final.
- Quiz: two quizzes each week, two drop
- Homework: collected twice a week, partially graded, two drop
- MatLab: totally three assignment
- Grade Distribution

MatLab	5
Homework	10
Quiz	15
Midterm	25
Final Exam	45

Please read a syllabus on Sakai carefully.

Overview of the Course

- Chapt. 1 First-order Differential Equations(DE)
- Chapt. 2 First-order Systems
- Chapt. 3 Linear Systems
- Chapt. 4 Forcing and Resonance
- Chapt. 5 Nonlinear Systems

Overview of Chapt. 1

- We want to predict the future. Why? ▸ Tacoma Bridge
- Differential Equations (DE) are powerful modeling tools for this purpose.
- To analyze DE, there are three basic types of techniques:

Overview of Chapt. 1

- We want to predict the future. Why? ▸ Tacoma Bridge
- Differential Equations (DE) are powerful modeling tools for this purpose.
- To analyze DE, there are three basic types of techniques:
  - **Analytic**: finding formulas
  - **Qualitative**: a rough sketch/long-term behavior
  - **Numerical**: arithmetic that yields approximations of solutions

([PRG] page 1)

Sect. 1.1 Modeling via DEs

What is a Model?

- The basic steps for model building

Step 1 State the *assumptions*.

Step 2 Describe the *variables* and *parameters*.

Step 3 Use the *assumptions* to derive equations relating the *quantities*.

Sect. 1.1 Modeling via DEs

What is a Model?

- The basic steps for model building

Step 1 State the *assumptions*.

Step 2 Describe the *variables* and *parameters*.

Step 3 Use the *assumptions* to derive equations relating the *quantities*.

The quantities fall into three basic categories

- **independent variable**: quantity that does not depend on any other quantities
- **dependent variables**: quantity that depends on independent variables
- **parameters**: quantity that does not depend on independent variable, but that can be adjusted.

([PRG] page 2,3)

Example) Unlimited Population Growth

Assumption

The rate of growth of the population is proportional to the size of the population.

Example) Unlimited Population Growth

Assumption

The rate of growth of the population is proportional to the size of the population.

Quantities

- t=time (independent variable)
- P=population (dependent variables),
- k=proportionality constant (parameter) between the rate of growth of the population and the size of the population.

Example) Unlimited Population Growth

Assumption

The rate of growth of the population is proportional to the size of the population.

Quantities

- t=time (independent variable)
- P=population (dependent variables),
- k=proportionality constant (parameter) between the rate of growth of the population and the size of the population.

Equation

$$\frac{dP}{dt} = kP$$

([PRG] page 4,5)

What does the model predict?

Unlimited Population Growth

$$\frac{dP}{dt} = kP$$

What if

- $P \equiv 0$[i] (Detail 1)[ii]
- $P(t_0) > 0$ for some time t_0 (Detail 2)

[i] This notation means the function P is identically equal to 0.
[ii] We will discuss detail in class on blackboard.

So the graph might look like

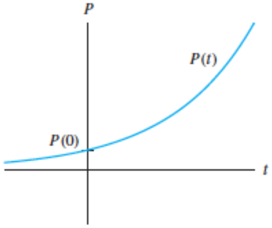


Figure 1.2

Qualitative analysis: population explosions happen as long as $P(0) > 0$.
([PRG] page 5, 6)

How about analytic method?

- Given constants $k > 0$ and P_0, consider the problem

$$\frac{dP}{dt} = kP, \quad P(0) = P_0,$$

  which is called an **initial-value problem.**
- A function that satisfies the equations is called a **solution**.

How about analytic method?

- Given constants $k > 0$ and P_0, consider the problem

$$\frac{dP}{dt} = kP, \quad P(0) = P_0,$$

  which is called an **initial-value problem.**
- A function that satisfies the equations is called a **solution**.
- Note

$$P(t) = P_0 e^{kt}$$

  is a solution to the initial value problem (Detail 3).
- This way of solving DE is called an **analytic** technique.

([PRG] page 7)

The U.S Population

How can this model be used? The U.S. Population (Detail 4)

Table 1.1
U.S. census figures, in millions of people (see www.cen

Year	t	Actual	$P(t) = 3.9e^{0.03067t}$
1790	0	3.9	3.9
1800	10	5.3	5.3
1810	20	7.2	7.2
1820	30	9.6	9.8
1830	40	13	13
1840	50	17	18
1850	60	23	25
1860	70	31	33
1870	80	39	45
1880	90	50	62
1890	100	63	84
1900	110	76	114
1910	120	91	155
1920	130	106	210

To adjust the previous model, we add

Assumptions

- If the population is small, the rate of growth of the population is proportional to its size
- If the population is too large, the population will decrease.

To adjust the previous model, we add

Assumptions

- If the population is small, the rate of growth of the population is proportional to its size
- If the population is too large, the population will decrease.

Quantities

t=time, P=population, k=growth-rate coefficient for small populations,

To adjust the previous model, we add

Assumptions

- If the population is small, the rate of growth of the population is proportional to its size
- If the population is too large, the population will decrease.

Quantities

t=time, P=population, k=growth-rate coefficient for small populations, N=carrying capacity

To adjust the previous model, we add

Assumptions

- If the population is small, the rate of growth of the population is proportional to its size
- If the population is too large, the population will decrease.

Quantities

t=time, P=population, k=growth-rate coefficient for small populations, N=carrying capacity

Equations (Detail 5)

$$\frac{dP}{dt} = k\left(1 - \frac{P}{N}\right)P.$$

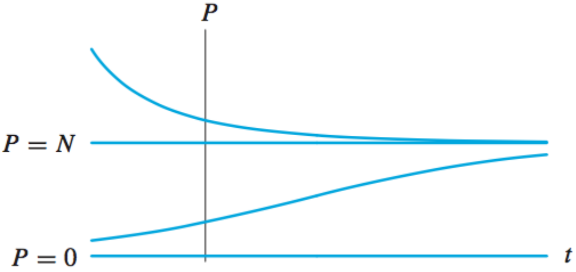
This model is called the **logistic population model**. ([PRG] page 9,10)

Qualitative Analysis

(Detail 6)

Qualitative Analysis

(Detail 6)



Predator-Prey Systems

Assumption

- If no foxes are present, the rabbits reproduce at a rate proportional to their population.

Predator-Prey Systems

Assumption

- If no foxes are present, the rabbits reproduce at a rate proportional to their population.
- The rate at which the rabbits are eaten by foxes is proportional to the rate at which the foxes and rabbits interact.

Predator-Prey Systems

Assumption

- If no foxes are present, the rabbits reproduce at a rate proportional to their population.
- The rate at which the rabbits are eaten by foxes is proportional to the rate at which the foxes and rabbits interact.
- Without rabbits, the fox population declines at a rate proportional to itself.

Predator-Prey Systems

Assumption

- If no foxes are present, the rabbits reproduce at a rate proportional to their population.
- The rate at which the rabbits are eaten by foxes is proportional to the rate at which the foxes and rabbits interact.
- Without rabbits, the fox population declines at a rate proportional to itself.
- The rate at which foxes are born is proportional to the number of rabbits eaten by foxed.

Predator-Prey Systems

Assumption

- If no foxes are present, the rabbits reproduce at a rate proportional to their population.
- The rate at which the rabbits are eaten by foxes is proportional to the rate at which the foxes and rabbits interact.
- Without rabbits, the fox population declines at a rate proportional to itself.
- The rate at which foxes are born is proportional to the number of rabbits eaten by foxed.

(Detail 7) ([PRG] page 12,13)

Quantities

t = time, F = population of fox, R = population of rabbit

α = growth-rate coefficient of rabbits

β = constant of proportionality that measures the number of rabbit-fox interactions

γ = death-rate coefficient of foxes

δ = constant of proportionality that measures the benefit to the fox population of an eaten rabbit

([PRG] page 12, 13)

Equations

$$\frac{dR}{dt} = \alpha R - \beta RF$$
$$\frac{dF}{dt} = -\gamma F + \delta RF.$$

This pair of equations is also called a **first-order system** of ordinary differential equations (ODE). We will learn more about this model in Chapter 2.
([PRG] page 13)

The Basic Approaches (Quick Review)

- Analytic: finding explicit formulas of solutions
- Qualitative: a rough sketch/long-term behavior
- Numerical: arithmetic that yields approximations of solutions (not covered in this seciton).

([PRG] page 14)

Quick Summary

1. Chapt. 1 First-Order Differential Equations
2. 1.1 Modeling via DEs
    - What is a Model?
    - Unlimited Population Growth
    - Limited Resources and the Logistic Population Model
    - Predator-Prey Systems

Quick Summary

1. Chapt. 1 First-Order Differential Equations
2. 1.1 Modeling via DEs
    - What is a Model?
    - Unlimited Population Growth
    - Limited Resources and the Logistic Population Model
    - Predator-Prey Systems

What is next

- 1.2 Analytic technique: Separation of Variables

Homework & Quiz

- **Homework** Exercises for Section 1.1: 1, 3, 5, 11, 13, 21(a)
- Write down your solutions and submit it in class. (You don't have to write down instructions of the problems. Save your time!)
- Due date will be announced on Sakai.
- For Problem 11, the unit of time t is a year. For example, $t = 0$ is the year 2000 and $t = 1$ is the year 2001.
- If you have any questions regarding homework problem, please feel free to come to my office hour. (Office hour info is posted at Sakai.)
- Take a course survey.
  https://goo.gl/forms/qEMUJYpspIbfajQ93