Chapter 1 First Order Differential Equations

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Overview

What is a DE and What Is a Solution?

Form for a 1st-order DE

$$\frac{dy}{dt} = f(t, y)$$

Solution to the DE

A function $y(t)$ is a **solution** to the DE if it satisfies the equation

$$\frac{dy}{dt} = f(t, y(t)) \quad \text{for all } t.$$

e.g.) Consider

$$\frac{dy}{dt} = y.$$

Check that $y_1(t) = e^{3t}$ is a solution whereas $y_2(t) = \sin t$ is not a solution. (Detail 1) ([PRG] page 21)

Checking a given function is a solution to a given equation.

e.g.)Consider

$$\frac{dy}{dt} = \frac{y^2 - 1}{t^2 + 2t}$$

Check $y_1(t) = 1 + t, y_2(t) = 1 + 2t$. (Detail 2)
([PRG] page 22)

Initial-Value Problems and the General Solution

Form of an initial value problem

Given DE with an **initial condition**, that is

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0,$$

find a solution $y(t)$ (satisfying the DE and the initial condition)

e.g.) Consider an initial-value problem

$$\frac{dy}{dt} = 12t^3 - 2\sin t, \quad y(0) = 3.$$

([PRG] page 23)

What is a general solution?

- Continuing to consider the previous example,

$$y(t) = 3t^4 + 2\cos t + c$$

  solves the given equation where c is a constant of integration. (Detail 3)
- The solution $y_1(t)$ is called the **general solution** to the given equation because we can use it to solve any initial-value problem. (Detail 4)

([PRG] page 24)

Separable Equations

In general, it is hard to find explicit formula for solutions of a DE. But for some special type of DE, we might be able to find one.

Separable DE

A DE is called **separable** if it is in the form

$$\frac{dy}{dt} = g(t)h(y).$$

e.g.) Consider

$$1)\ \frac{dy}{dt} = yt, \qquad 2)\ \frac{dy}{dt} = y + t, \qquad 3)\frac{dy}{dt} = \frac{t+1}{ty+t}$$

1) separable, 2) not separable 3) separable. (Detail 5)
([PRG] page 24, 25)

Autonomous DE

A specific type of separable equations is in the form

$$\frac{dy}{dt} = h(y).$$

This type of DE is said to be **autonomous**.
e.g.) the logistic equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right).$$

([PRG] page 24, 25)

Solve separable DE (Separation of Variables)

Consider

$$\frac{dy}{dt} = \frac{t}{y^2}.$$

Informally,

$$y^2 dy = t dt.$$

Integrating both sides,

$$\int y^2 dy = \int t dt,$$

so

$$\frac{y^3}{3} = \frac{t^2}{2} + c, \quad \text{that is,} \quad y(t) = \left(\frac{3t^2}{2} + 3c\right)^{1/3}.$$

This is informal computation, but for the real story, see ([PRG] page 25, 26, 27) For more examples, see exercises. (Detail 6)

Getting Stuck

- Consider
$$\frac{dy}{dt} = \frac{y}{1+y^2}.$$
Then
$$\ln|y| + \frac{y^2}{2} = t + c.$$
(Detail 7) Sometimes, there might be no explicit formula for the general solution.
- Consider
$$\frac{dy}{dt} = \sec(y^2).$$
(Detail 8) Sometimes, it might be impossible to perform the necessary integration.

([PRG] page 28, 29)

Application) Savings Model

- We deposit $ 5000 with 2% interest compounded continuously
- Ater 10 years, we withdraw $ 500 each year.

How long will this money last? (Detail 9)
([PRG] page 29-31)

Application) A Mixing Problem

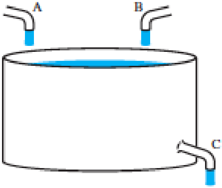

Figure 1.9
Mixing vat.

(Detail 10)
([PRG] page 32, 33)

- The vat contains 100 gallons of liquid. And Flowing in=Flowing out.
- The vat is kept well mixed.

And

- Sugar water (5 tablespoons of sugar/gallon) enters the vat through pipe A (2 gallons/minute).
- Sugar water(10 tablespoons of sugar/gallon) enters the vat through pipe B (1 gallon/minute)
- Sugar water leaves the vat through pipe C at (3 gallons/minute)

Overview

- Homework Exercises for Section 1.2: 1, 5-9 odd, 25-29 odd, 39
- Write down your solutions and submit it in class.
- Due date will be announced on Sakai.
- If you have any questions regarding homework problem, please feel free to come to my office hour. (Office hour info is posted at Sakai.)