Chapter 1 First Order Differential Equations

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Overview

1. 1.3 Qualitative Technique: Slope Fields
    - The Geometry of $dy/dt = f(t, y)$
    - Slope Fields
    - Important Special Cases
    - Analytic versus Qualitative Analysis
    - An RC Circuit
    - homework

The Geometry of $dy/dt = f(t, y)$

If y is a solution to $dy/dt = f(t, y)$ and $y(t_1) = y_1$, geometrically the equations means the slope of the tangent line to the graph of $y(t)$ at (t_1, y_1) is given by the number $f(t_1, y_1)$.

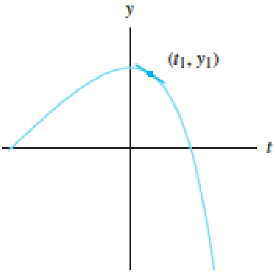

Figure 1.10

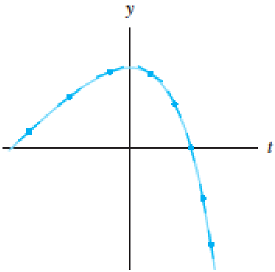

Figure 1.11

([PRG] page 36)

Slope Fields

Consider

$$\frac{dy}{dt} = y - t.$$

Then RHS(the right hand side) is given by the function $f(t, y) = y - t$.

Table 1.2

Selected slopes corresponding to the differential equation $dy/dt = y - t$

(t, y)	$f(t, y)$	(t, y)	$f(t, y)$	(t, y)	$f(t, y)$
$(-1, 1)$	2	$(0, 1)$	1	$(1, 1)$	0
$(-1, 0)$	1	$(0, 0)$	0	$(1, 0)$	-1
$(-1, -1)$	0	$(0, -1)$	-1	$(1, -1)$	-2

([PRG] page 37)

A "Sparse" Slope Field

Based on the table, we can draw each mini-tangent line whose slope is $f(t, y)$

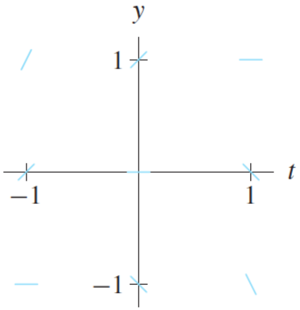

Figure 1.13

([PRG] page 37, 38)

A Computer-Generated Version

This sketch is called **slope field**. We can compare this with general solutions $y(t) = t + 1 + ce^t$. (Detail 1)

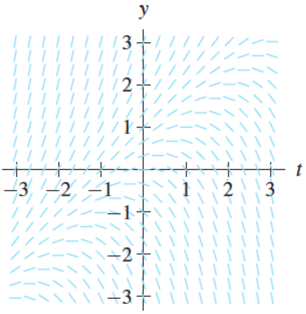

Figure 1.14

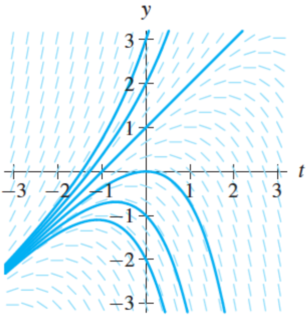

Figure 1.15

([PRG] page 39)

Important Special Cases

For the equation $dy/dt = f(t)$, RHS is solely a function of t. Geometrically, all of the slope marks on each vertical line are parallel.

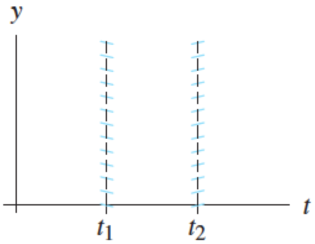

Figure 1.16

([PRG] page 39, 40)

Example

e.g.) Consider

$$\frac{dy}{dt} = 2t.$$

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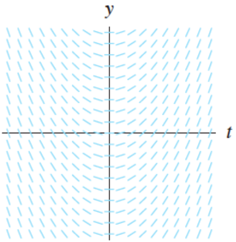


Figure 1.17

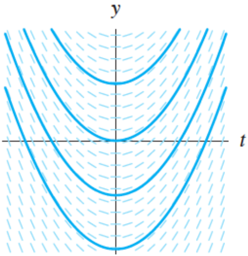
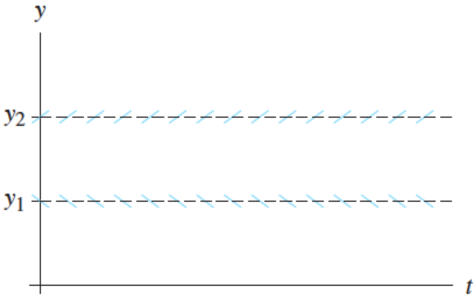


Figure 1.18

(Detail 2) ([PRG] page 40)

Slope Fields for Autonomous Equations

In the case of $dy/dt = f(y)$, the slope field is parallel along each horizontal line.



([PRG] p. 40)

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$$\frac{dy}{dt} = 4y(1 - y). \qquad \text{(Detail 3)}$$

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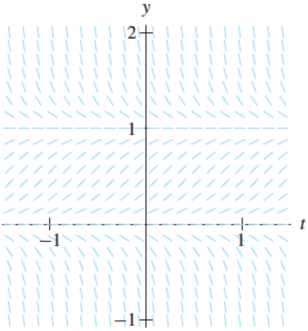

Figure 1.20

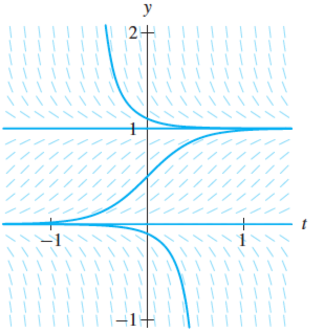

Figure 1.21

Analytic versus Qualitative Analysis

Consider

$$\frac{dy}{dt} = e^{y^2/10} \sin^2 y.$$

To apply separation of variables, we have to evaluate

$$\int \frac{dy}{e^{y^2/10} \sin^2 y} = \int dt,$$

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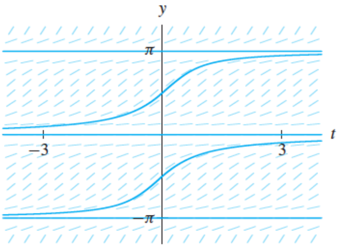
which is difficult.
([PRG] page 42)

Analytic versus Qualitative Analysis

So we resort to qualitative methods. (Detail 4)



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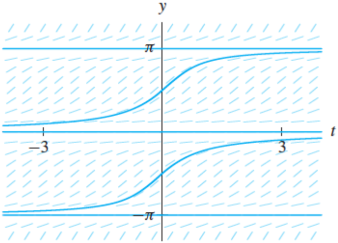

But Remaining Question: the graphs do not cross the horizontal lines?

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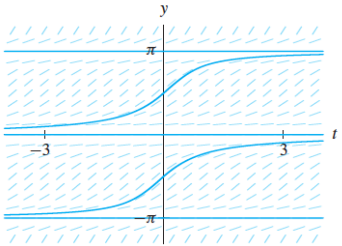

But Remaining Question: the graphs do not cross the horizontal lines?
(It will be covered in Sect. 1.5. Existence and Uniqueness of Solutions)

An RC Circuit

Quantities

$V(t)$: input voltage, $v_c(t)$: voltage across the capacitor,
R, C: positive parameters

Equation

$$\frac{dv_c}{dt} = \frac{V(t) - v_c}{RC}.$$

We will draw slope fields for 1) constant voltage source, 2) On-Off voltage source.

Constant Voltage Source

Suppose $V(t)$ is a constant K for all t. Then the equation is

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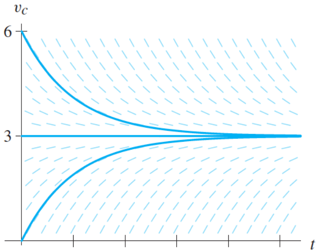
We can draw slope field.



(It is drawn with the choice of $R = 0.5, C = 1, K = 3$.) ([PRG] p. 45)

On-Off voltage source

Suppose $V(t) = K > 0$ for $0 \leq t < 3$, but at $t = 3$, this voltage is turned off. Our DE is

$$\frac{dv_c}{dt} = \begin{cases} \frac{K - v_c}{RC} & \text{for } 0 \leq t < 3; \\ \frac{-v_c}{RC} & \text{for } t > 3. \end{cases}$$

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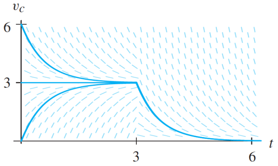
Then the slope field is



(with the choice of $R = 0.5$, $C = 1$, $K = 3$.) ([PRG] p.46)

Overview

- Homework exercises: 16, 21
- Final answers to 16 will be posted on Sakai.
- There will be MatLab assigned about this section. I will announce this and upload materials on Sakai.