Chapter 1 First Order Differential Equations

Sect. 1.4 Numerical Technique: Euler's Method

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Overview

Given

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0,$$

suppose we want to find quantitative information about solutions.

- We can draw a slope field (qualitative), but *it does not give us quantitative information*.
- Analytic method can give us quantitative information, but *finding an explicit formula is difficult most of time.*
- However, numerical methods provide *quantitative* information even if we cannot find their formula!

([PRG] p. 52)

Stepping along the Slope Field

Begin with

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0.$$

We want to find quantitative information.

The idea of Euler's method

1. Start at the point (t_0, y_0) in the slope field
2. Take tiny steps dictated by the tangents in the slope field.

([PRG] p. 52)

Euler's method

Consider $\frac{dy}{dt} = f(t, y)$, $\quad y(t_0) = y_0$. Choose a (small) **step size** Δt. Start at (t_0, y_0). Take

$$t_1 = t_0 + \Delta t$$
$$y_1 = y_0 + f(t_0, y_0)\Delta t. \quad \text{(Detail 1)}$$

And continue this process.

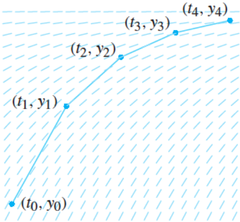


Figure 1.31

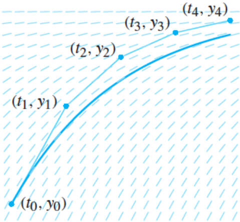


Figure 1.32

Euler's method

Euler's method for $\frac{dy}{dt} = f(t, y)$

Given an initial condition $y(t_0) = y_0$ and the step size Δt, compute (t_{k+1}, y_{k+1}) form (t_k, y_k) as follows

1. Compute the slope $f(t_k, y_k)$
2. Calculate

$$t_{k+1} = t_k + \Delta t$$
$$y_{k+1} = y_k + f(t_k, y_k)\Delta t. \quad \text{(Detail 2)}$$

([PRG] p.54)

Example

Consider

$$\frac{dy}{dt} = 2y - 1, \quad y(0) = 1.$$

Goal: to evaluate $y(1)$.
Separating and integrating, we obtain

$$y(t) = \frac{e^{2t} + 1}{2}.$$

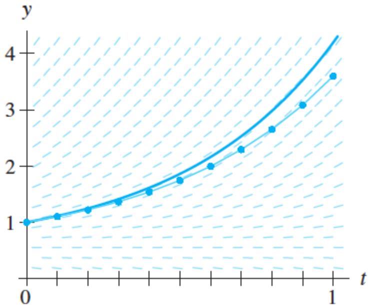
So,

$$y(1) = \frac{e^2 + 1}{2} \approx 4.195$$

([PRG] p.55)

Example

We can also apply Euler's method with $\Delta t = 0.1$ to obtain (Detail 3)



Euler methods yields $y(1) \approx 3.596$ whereas analytic methods yields $y(1) \approx 4.195$. ([PRG] p.56, 57)

Example

- To improve our approximation, we take a smaller step, $\Delta t = 0.05$.
- Usually we get a better approximation:

$$y(1) \approx 3.864$$

- Price: More computation must be done to approximate the solution at $t = 1$.

([PRG] p.56, 57)

Example

Method	**Approximation of $y(1)$**
Analytic	$y(1) = \frac{e^2+1}{2} \approx 4.195$
Euler with $\Delta t = 0.1$	$y(1) \approx 3.596$
Euler with $\Delta t = 0.05$	$y(1) \approx 3.864$
Euler with $\Delta t = 0.01$	$y(1) \approx 4.1223$

Table: Better approximation with a smaller step

([PRG] p.55, 56, 57)

An RC Circuit with Periodic Input

Consider

$$\frac{dv_c}{dt} = \frac{V(t) - v_c}{RC}.$$

Take $R = 0.5$, $C = 1$, $V(t) = \sin(2\pi t)$. Then

$$\frac{dv_c}{dt} = -2v_c + 2\sin(2\pi t)$$

We apply Euler's method to get

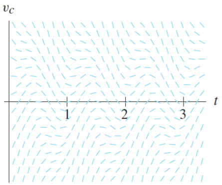


Figure 1.39

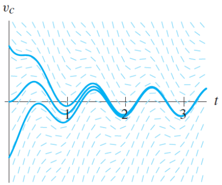


Figure 1.40

Erros in Numerical Methods

- When we apply Euler's method, we always make an error.
- Sometimes, it leads to disastrously wrong approximations.
- Consider

$$\frac{dy}{dt} = e^t \sin y.$$

- If we apply Euler's method.....

([PRG] p.60)

Erros in Numerical Methods

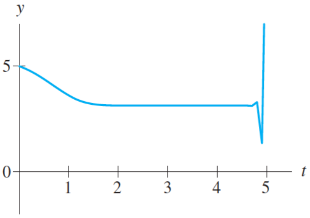


Figure 1.41
Euler's method applied to

$$\frac{dy}{dt} = e^t \sin y$$

with $\Delta t = 0.1$

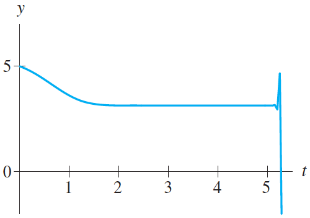


Figure 1.42
Euler's method applied to

$$\frac{dy}{dt} = e^t \sin y$$

with $\Delta t = 0.05$.

Question: are we sure this approximation is wrong?
(It will be covered in Sect. 1.5 Existence and Uniqueness of Solutions) ([PRG] p.61)

- We have the analytic, the numeric, and the qualitative approaches.
- Which method is the best depends both
  - on the DE in question and
  - on what we want to know about the solutions,

Overview

1. 1.4 Numerical Technique: Eulers Method
    - Euler's method
    - Example
    - An RC Circuit with Periodic Input
    - Erros in Numerical Methods
    - The Big Three
    - homework

What's next: Sect. 1.5 Existence and Uniqueness of Solutions

- No homework for this section.
- Instead, there will be a MatLab assignment. It will be announced on Sakai.

References

Paul Blanchard, Robert L. Devaney, Glen R. Hall
Differential Equations, fourth edition.