Chapter 1 First Order Differential Equations

Sect. 1.5 Existence and Uniqueness of Solutions

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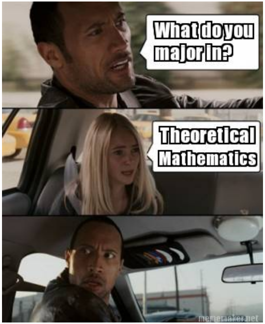


Overview

1. 1.5 Existence and Uniqueness of Solutions
   - What Does It Mean to Say Solutions Exists?
   - Existence
   - Extendability
   - Uniqueness
   - Applications of the Uniqueness Theorem
   - Uniqueness and Numerical Approximation
   - homework

What does it mean to say solutions exists?

Consider the algebraic equation

$$2x^5 - 10x + 5 = 0.$$

- Even though we do not know how to solve it,
- we know there must be a value of x for which the equation is satisfied. (Detail 1)

Given DE, without computing solutions or drawing graphs, we can discuss the existence of solutions
([PRG] p. 63, 64)

Existence

Theorem

Suppose $f(t, y)$ is a continuous function in a rectangle in the ty-plane.
If (t_0, y_0) is a point in this rectangle,
then there exists an $\varepsilon > 0$ and a function $y(t)$ defined for $t_0 - \varepsilon < t < t_0 + \varepsilon$ that solve the initial-value problem

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0.$$

(Detail 2) ([PRG] p.64)

Extendability

Why solutions only exist very short interval of time?
Consider

$$\frac{dy}{dt} = 1 + y^2, \quad y(0) = 0.$$

Separating and integrating,

$$y(t) = \tan t. \quad \text{(Detail 3)}$$

([PRG] p.65)

Extendability

The solution blows up at a finite time!

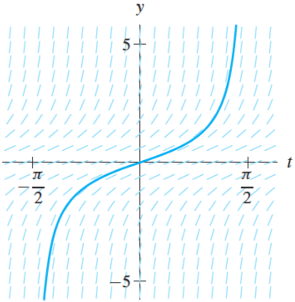


Figure 1.44

Therefore, the restriction that a solution only exists for short time interval is necessary.

Uniqueness

Theorem (Uniqueness)

Suppose $f(t, y)$ and $\partial f/\partial y$ are continuous functions in a rectangle in the ty-plane. If (t_0, y_0) is a point in this rectangle and if $y_1(t)$ and $y_2(t)$ are two functions that solve the initial-value problem

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0$$

for all t in the interval $t_0 - \varepsilon < t < t_0 + \varepsilon$, then

$$y_1(y) = y_2(t)$$

for $t_0 - \varepsilon < t < t_0 + \varepsilon$.

([PRG] p. 66)

Lack of Uniqueness

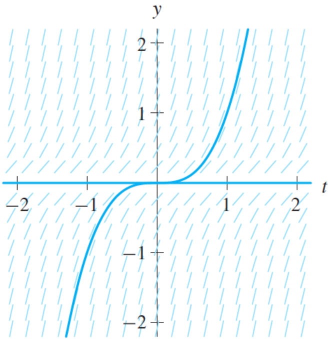
Consider

$$\frac{dy}{dt} = 3y^{2/3}, \quad y(0) = 0.$$

We have two solutions:

$$y_1(t) = 0, \quad y_2(t) = t^3.$$

(Detail 4)



Applications of the Uniqueness Theorem

Uniqueness Theorem says

> *"If two solutions are ever in the same place at the same time, then they are the same function."*

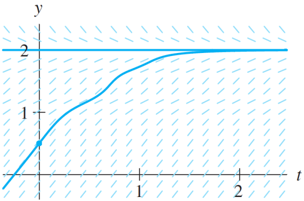
This form of the Uniqueness Theorem is very valuable as the following example show.
([PRG] p.68)

Role of Equilibrium Solutions

Consider

$$\frac{dy}{dt} = \frac{(y^2 - 4)(\sin^2 y^3 + \cos y - 2)}{2}, \quad y(0) = \frac{1}{2}$$

We can draw a slope field



The Uniqueness Theorem implies they do not cross!

Comparing Solutions

Consider

$$\frac{dy}{dt} = \frac{(1+t)^2}{(1+y)^2}.$$

Suppose $y_2(t)$ is the solution satisfying $y(0) = -0.1$.

- What do we know about y_2?
- Claim: $y_2(t) \leq t$
- Why? compare y_2 to $y_1(t) = t$ by using the Uniqueness Theorem (Detail 5)

([PRG] p. 69)

Uniqueness and qualitative analysis

Consider

$$\frac{dy}{dt} = (y-2)(y+1).$$

- Set $f(y) = (y-2)(y+1)$. Note $f(2) = f(-1) = 0$. So $y = 2, -1$ are equilibrium solutions
- By Uniqueness, any solution $y(t)$ with $-1 < y(0) < 2$ must also satisfies
$$-1 < y(t) < 2 \quad \text{for all } t.$$
- If a solution y satisfies $-1 < y(0) < 2$, then y is decreasing for all t. (Detail 7)

Uniqueness and qualitative analysis

The previous observations are consistent with the slope field

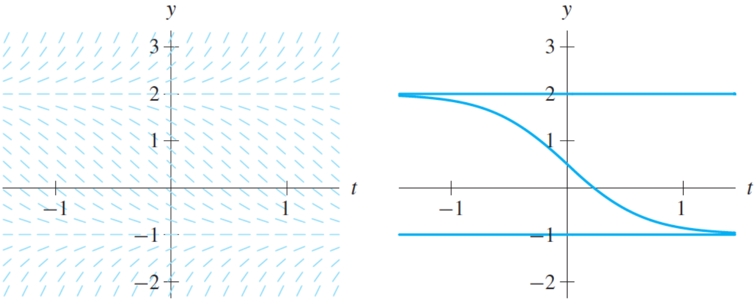


Figure 1.48 Figure 1.49

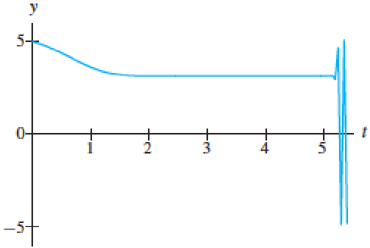
But how do we make sure the solutions converges to -1 as $t \to \infty$? (Detail 8)

Uniqueness and Numerical Approximation

Consider

$$\frac{dy}{dt} = e^t \sin y, \quad y(0) = 5.$$

Euler's method yields



This is wrong because $y(t) = n\pi$ is a solution for any integer n.
([PRG] p. 69)

Overview

What's next: Sect. 1.6 Equilibria and the Phase Line

homework

- Homework exercises: 1, 3, 5, 11, 13.
- Due date will be announced on Sakai.

References

Paul Blanchard, Robert L. Devaney, Glen R. Hall
Differential Equations, fourth edition.