Chapter 1 First Order Differential Equations

Sect. 1.6 Equilibria and the Phase Line

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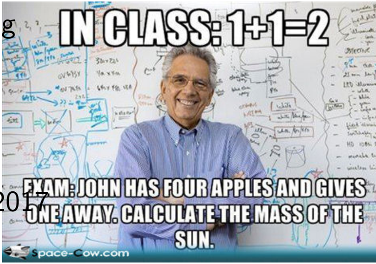


Overview

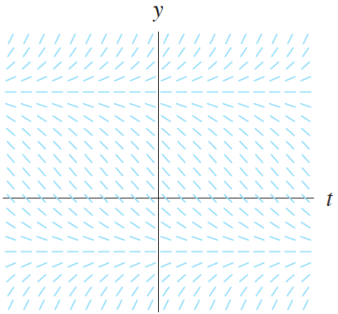
Autonomous Equations

- In this section, we only consider autonomous equations of the form

$$\frac{dy}{dt} = f(y).$$

- For autonomous equations, slope marks are parallel along horizontal lines.

([PRG] p.74,75)



Autonomous Equation

- Therefore, if we know the slop field along a single vertical line $t = t_0$, then we know the slop field in the entire ty-plane.
- So instead of drawing the entire slope field, we draw just one line. This line is called the **phase line** for the autonomous equations.

([PRG] p.75)

Phase Line of a Logistic Equation

As an example, consider

$$\frac{dy}{dt} = (1 - y)y.$$

Set $f(y) = (1 - y)y$.

Then the phase line is (Detail 1)

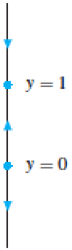


Figure 1.52
Phase line for
$dy/dt = (1 - y)y$.

Phase Line of a Logistic Equation

Compare to the slope field and the graphs. (Detail 2)

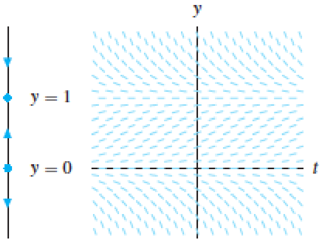


Figure 1.53
Phase line and slope field of
$dy/dt = (1 - y)y$.

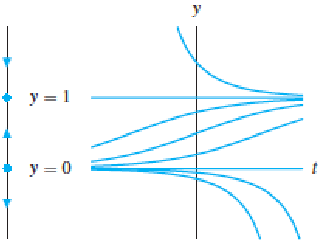


Figure 1.54
Phase line and sketches of the graphs of
solutions for $dy/dt = (1 - y)y$.

How to Draw Phase Lines

1. Draw the y-line
2. Find the equilibrium points and mark them on the line
3. Find the intervals of y-values for which $f(y) > 0$, and draw arrows pointing up in these intervals
4. Find the intervals of y-values for which $f(y) < 0$, and draw arrows pointing down in these intervals.

([PRG] p.76)

Examples

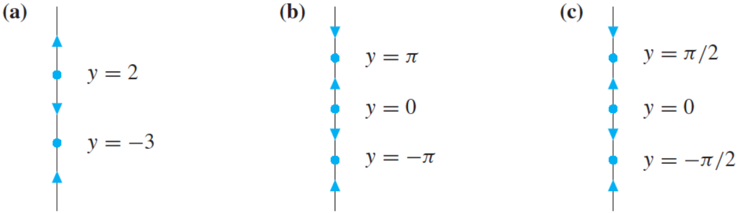
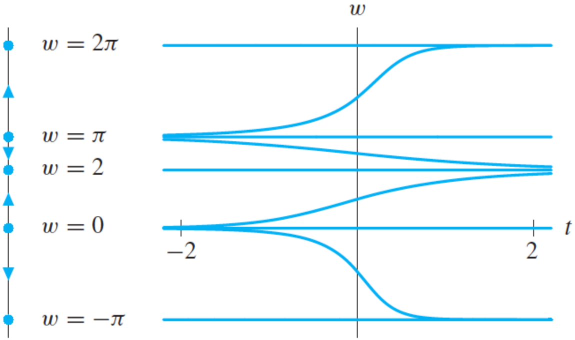


Figure 1.55
Phase lines for **(a)** $dy/dt = (y-2)(y+3)$, **(b)** $dy/dt = \sin y$, and **(c)** $dy/dt = y \cos y$.

(Detail 3) ([PRG] p.77)

How to Use Phase Lines to Sketch Solutions

We can obtain rough sketches of the graphs of solutions directly from the phase lines. Consider $dw/dt = (2 - w)\sin w$. (Detail 4)



How to Use Phase Lines to Sketch Solutions

Suppose $y(t)$ is a solution to an autonomous equation $dy/dt = f(y)$.

- If $f(y(0)) = 0$, then $y(0)$ is an equilibrium point and $y(t) = y(0)$ for all t
- If $f(y(0)) > 0$, then $y(t)$ is increasing for all t and either $y(t) \to \infty$ as t increases or $y(t)$ tends to the first equilibrium point larger than $y(0)$
- If $f(y(0)) < 0$, then $y(t)$ is decreasing for all t and either $y(t) \to -\infty$ as t increases or $y(t)$ tends to the first equilibrium point smaller than $y(0)$.

Warning: Not All Solutions Exists for All Time

Consider $\frac{dy}{dt} = (1+y)^2$. We can sketch the graphs.

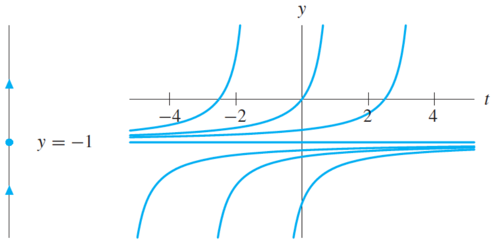


Figure 1.61

Question: do they blow up in finite time? We can't tell.

Warning: Not All Solutions Exists for All Time

Separating variables,

$$y(t) = -1 - \frac{1}{t+c}$$

Therefore, it blows up in finite time. (Detail 5)

Warning: Not All Solutions Exists for All Time

Another dangerous example is $\frac{dy}{dt} = \frac{1}{1-y}$. We can sketch the graphs.

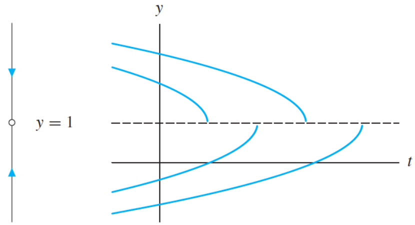
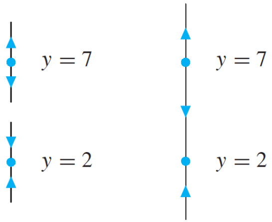


Figure 1.62

(Detail 6)

The Role of Equilibrium Points

1. For $\frac{dy}{dt} = g(y)$, the sign of g can change only at an equilibrium point.
2. So it is enough to know behavior near the equilibrium points.



([PRG] p. 83)

Classification of Equilibrium Points

An equilibrium point y_0 is called a **sink** if any solution with initial condition sufficiently close to y_0 is asymptotic to y_0 as t increase, and is called a **source** if all solutions that start sufficiently close to y_0 tend toward y_0 as t decreases.

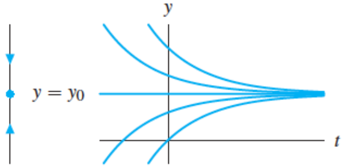


Figure 1.67
Phase line at a sink and graphs of solutions near a sink.

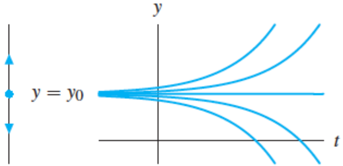


Figure 1.68
Phase line at a source and graphs of solutions near a source.

Classification of Equilibrium Points

Every equilibrium that is neither a source nor a sink is called a **node**.

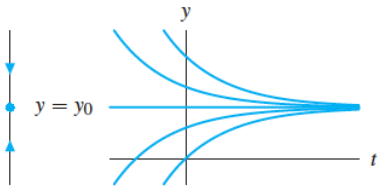


Figure 1.67
Phase line at a sink and graphs of solutions near a sink.

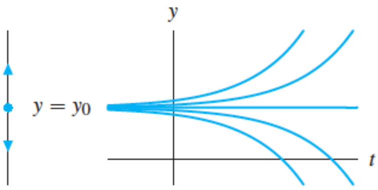


Figure 1.68
Phase line at a source and graphs of solutions near a source.

([PRG] p. 84, 85)

Identifying the type of an equilibrium point and "linearzation"

Theorem (Linearization)

Suppose y_0 is an equilibrium point of the differential equation $dy/dt = f(y)$ where f is a continuously differential function. Then,

- *if $f'(y_0) < 0$, then y_0 is a sink;*
- *if $f'(y_0) > 0$, then y_0 is a source; or*
- *if $f'(y_0) = 0$, then it is inconclusive.*

([PRG] p. 86)

Identifying the type of an equilibrium point and "linearzation"

To determine the type of an equilibrium point, the sign of the derivative at the point is helpful.

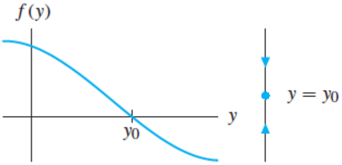


Figure 1.73
Phase line near a sink at $y = y_0$ for $dy/dt = f(y)$ and graph of $f(y)$ near $y = y_0$.

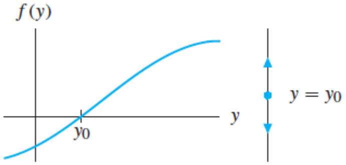


Figure 1.74
Phase line near a source at $y = y_0$ for $dy/dt = f(y)$ and graph of $f(y)$ near $y = y_0$.

Application) Modified Logistic Model

- For the pine squirrel to the Rocky Mountains,
- if the population is too small,
- fertile adults run the risk of not being able to find suitable mates,
- so again the rate of growth is negative.

([PRG] p. 87)

Modified Logistic Model

We can restate the assumptions succinctly:

Assumptions

- If the population is too big, the rate of growth is negative
- If the population is too small, the rate of growth is negative.
- If the population is zero, the growth rate is zero.

([PRG] p. 88)

Modified Logistic Model

Quantities

$$\begin{aligned} t &= \text{time (independent variable)} \\ S(t) &= \text{population of squirrels at time t (dependent variable)} \\ k &= \text{growth-rate coefficient (parameter)}, \\ N &= \text{carrying capacity (parameter)}, \textit{and} \\ M &= \text{"sparsity" constant (parameter)} \end{aligned}$$

Modified Logistic Model

To find an equation $dS/dt = g(S)$, note that we want

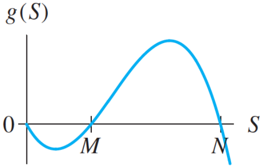


Figure 1.76
Graph of $g(S)$.

We modify the logistic model

$$\frac{dS}{dt} = kS\left(1 - \frac{S}{N}\right)(\text{something})$$

Consider what we want for g (Detail 7), we take

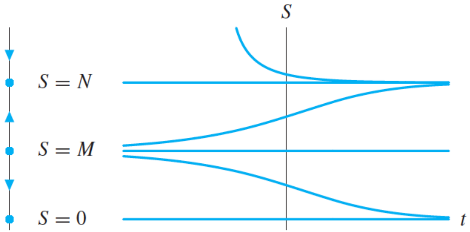
$$\frac{dS}{dt} = kS\left(1 - \frac{S}{N}\right)\left(\frac{S}{M} - 1\right)$$

Modified Logistic Model

How can we draw the graphs of solutions to our model?

$$\frac{dS}{dt} = kS\left(1 - \frac{S}{N}\right)\left(\frac{S}{M} - 1\right)$$

Because the equation is autonomous, we can use a phase line.



Overview

What's next: Sect. 1.7 Bifurcations

homework

- Homework exercises: 1-9 odd, 13-15 odd, 23, 33, 43
- Due date will be announced on Sakai
- Hint for 43: Consider Taylor's expansion of f, and explain why it works.

References

Paul Blanchard, Robert L. Devaney, Glen R. Hall
Differential Equations, fourth edition.