Chapter 1 First Order Differential Equations

Sect. 1.7 Bifurcations

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Overview

Equations with Parameters

- Recall the population model

$$\frac{dP}{dt} = kP$$

contains the parameter k. The behavior of solutions depends on the parameter.

- In general, how the behavior of solutions changes as the parameters vary is an important aspect of DE.

([PRG], p. 94)

Equations with Parameters

- We study autonomous equations with one parameter.
- A small change in the parameter usually results in only a small change in the nature of the solutions
- However, occasionally a small change in the parameter can lead to a *drastic change* in the long-term behavior of solutions.
- Such a change is called a **bifurcation**

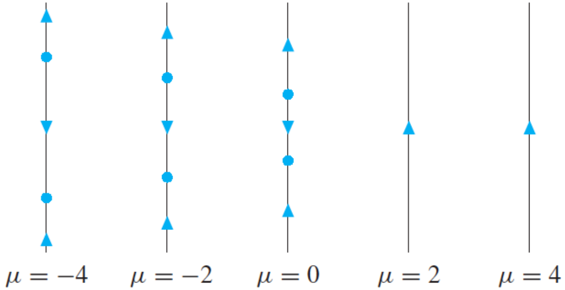
([PRG], p. 94)

Notation for DE depending on a parameter

- Consider
$$\frac{dy}{dt} = y^2 - 2y + \mu.$$
- Here, μ is a parameter, y is a dependent variable of t.
- We use a notation that distinguishes the dependence of the RHS on y and μ. Set
$$f_\mu(y) = y^2 - 2y + \mu.$$
- In general, we write
$$\frac{dy}{dt} = f_\mu(y)$$
This equation is called a **one-parameter family** of DE.

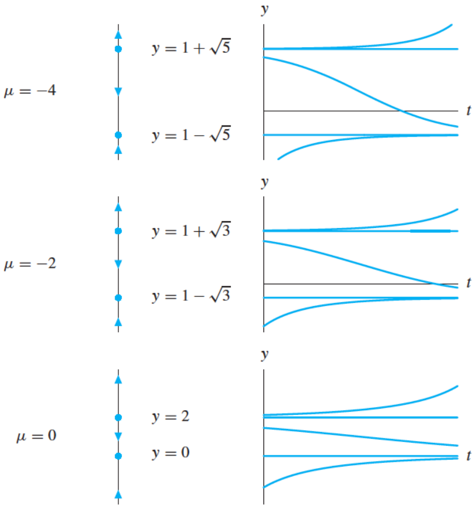
A One-Parameter Family with One Bifurcation

Continue to consider $\frac{dy}{dt} = f_\mu(y) = y^2 - 2y + \mu$. We can draw phase lines. (Detail 1)

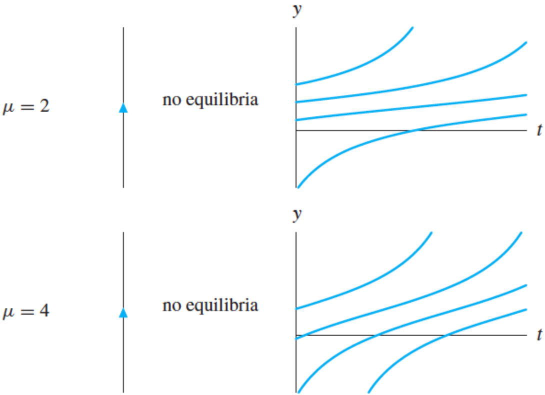


([PRG], p. 96)

A One-Parameter Family with One Bifurcation



A One-Parameter Family with One Bifurcation



A One-Parameter Family with One Bifurcation

To investigate the nature of this bifurcation, we draw the graphs of $f_\mu(y) = y^2 - 2y + \mu$ with different μ.

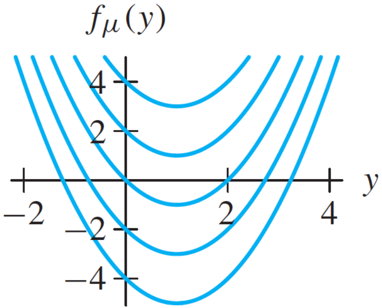


Figure 1.80

([PRG], p. 98)

A One-Parameter Family with One Bifurcation

A bifurcation occurs at $\mu = 1$ and $\mu = 1$ is called **bifurcation value**.

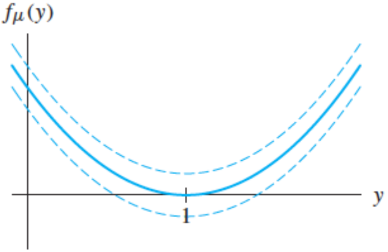


Figure 1.81

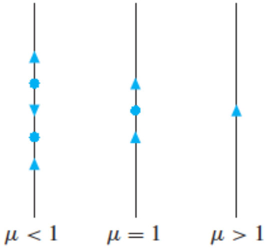
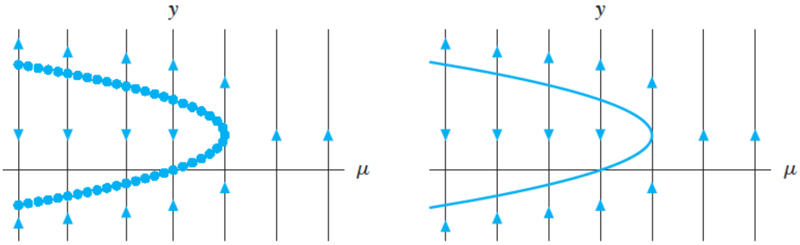


Figure 1.82

([PRG], p. 98)

The Bifurcation Diagram

A picture of the phase lines near a bifurcation is called a **bifurcation diagram**. (Detail 2)



([PRG], p. 99)

A bifurcation from one to three equilibria

Consider

$$\frac{dy}{dt} = g_\alpha(y) = y^3 - \alpha y = y(y^2 - \alpha). \quad \text{(Detail 3)}$$

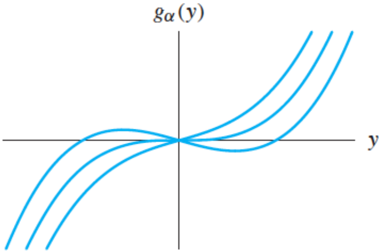


Figure 1.84

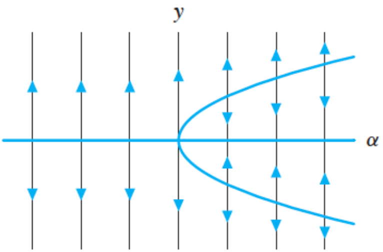


Figure 1.85

([PRG], p. 100)

Determining Bifurcation Values

Consider

$$\frac{dy}{dt} = f_\mu(y) = y(1-y)^2 + \mu.$$

We can draw phase lines based on

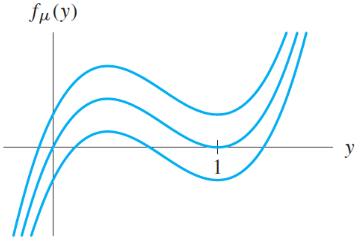


Figure 1.88
Graphs of

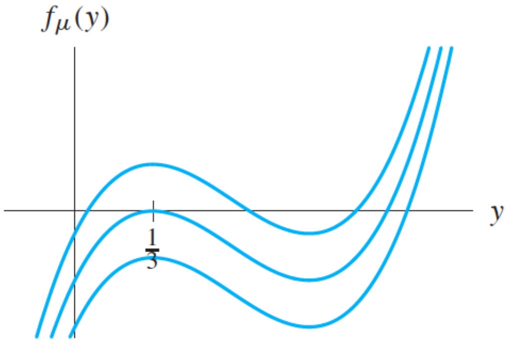
$$f_\mu(y) = y(1-y)^2 + \mu$$

for μ slightly greater than zero, μ equal to zero, and μ slightly less than zero.

(Detail 4)

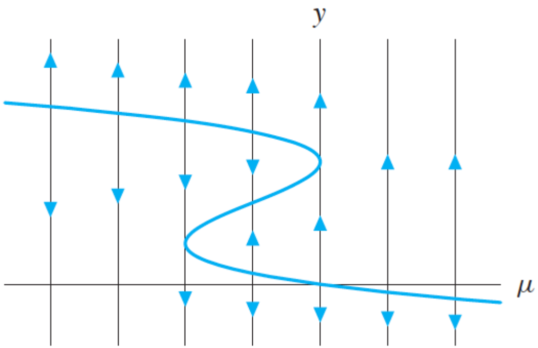
Determining Bifurcation Values

There is a second bifurcation. How can we find that? (Detail 5) It occurs when the graph of f_μ is tangent to the y-axis.



([PRG], p. 102)

Based on the previous observations, we can draw a bifurcation diagram: (Detail 6)



Sustainability

We can model the population of fish. What if we take fishing into account?

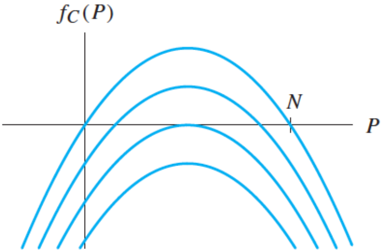
$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right) - C = f_C(P)$$

where fishing removes a certain constant number C of fish per season from the population.

Goal: how the behavior of P changes as C varies. (k, N are fixed.)
([PRG], p. 103)

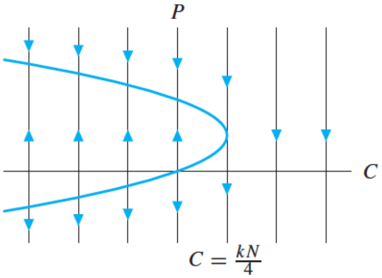
Sustainability

We can draw graphs of $f_C(P) = kP(1 - \frac{P}{N}) - C$ for several values for C.



([PRG], p. 104)

Eventually we can draw the bifurcation diagram. (Detail 7)



Conclusion: A slightly more fishing might cause extinction of fish.

Overview

1. 1.7 Bifurcations
   - Equations with Parameters
   - A One-Parameter Family with One Bifurcation
   - The Bifurcation Diagram
   - Determining Bifurcation Values
   - Sustainability
   - homework

What's next: Sect. 1.8 Linear Equations

homework

- Homework exercises: 1, 3, 9, 11, 15
- Due date will be announced on Sakai

References

Paul Blanchard, Robert L. Devaney, Glen R. Hall
Differential Equations, fourth edition.