Chapter 1 First Order Differential Equations

Sect. 1.8 Linear Equations

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Overview

- In Sect. 1.2, we developed an analytic method to separable equations,
- but by separating variables, we cannot even solve

$$\frac{dy}{dt} = y + t.$$

- In this and the next section, we develop two techniques for *linear* DE, (which is a generalization of the above example).

([PRG], p. 110)

Linear DE

A first-order DE is **linear** if it can be written in the form

$$\frac{dy}{dt} = a(t)y + b(t).$$

where $a(t), b(t)$ are arbitrary functions of t.

e.g.)

$$1)\frac{dy}{dt} = y+t, \quad 2)\frac{dy}{dt} = t^2y+\cos t, \quad 3)\frac{dy}{dt}-3y = ty+2, \quad 4)\frac{dy}{dt} = y^2$$

1) linear 2) linear, 3) linear, 4) not linear.
([PRG], p.110)

Additional Terminology for Linear Equations

For $\frac{dy}{dt} = a(t)y + b(t)$

- if $b(t) = 0$, then it is said to be **homogeneous** or *unforced*.
- Otherwise, it is said to be **nonhomogeneous**.

Linearity Principles

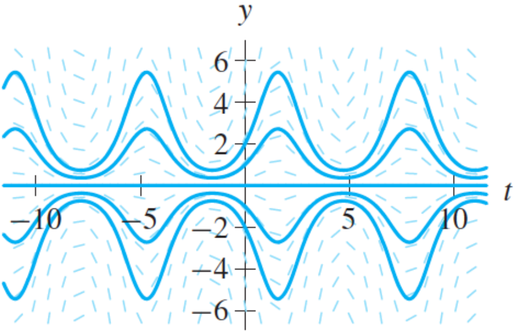
So far, we have not considered relations between solutions. But, for linear equations,

- the solutions to a linear equation are all related in a simple way.
- Given one or two nontrivial solutions, we get the rest by using the appropriate linearity principle.

([PRG], p.112)

The Homogeneous Case

Consider $\frac{dy}{dt} = (\cos t)y$.



Question: What relation do we have between solutions?

The homogeneous case

Linearity Principle

If $y_h(t)$ is a solution of the homogenous linear equation

$$\frac{dy}{dt} = a(t)y,$$

then any constant multiple of $y_h(t)$ is also a solution.
That is, $ky_h(t)$ is a solution for any constant k.

Why? (Detail 1)
Question) Are they all the solutions?

The Homogeneous Case

Yes they are all!

Linearity Principle (continued)

If y_h is a nontrivial solution of

$$\frac{dy}{dt} = a(t)y$$

and $a(t)$ is continuous, then ky_h is the general solution where k is any constant.

Why?

This slide is not contained in the textbook.

The Homogeneous Case

Roughly speaking, the Linearity Principle says

> *If y_h is a nontrivial solution to a homogeneous linear equation, then any constant multiple ky_h is a solution and they are all.*

The homogeneous case

Go back to $\frac{dy}{dt} = (\cos t)y$.

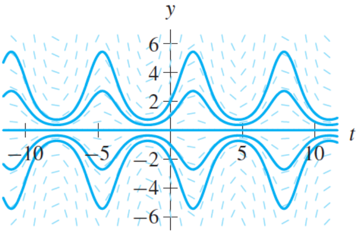


Figure 1.93
The slope field and graphs of various solutions to

$$\frac{dy}{dt} = (\cos t)y.$$

Note that the solutions are constant multiples of one another.

The Nonhomogeneous Case

Consider

$$\frac{dy}{dt} = y - 2.$$

Question) do we have the linearity principle?

The Nonhomogeneous Case

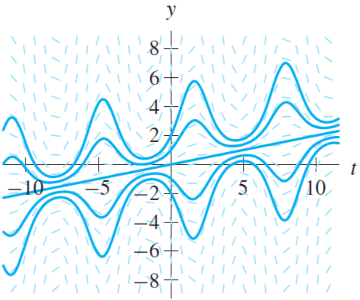


Figure 1.94
The slope field and graphs of various solutions to

$$\frac{dy}{dt} = (\cos t)y + \frac{1}{5}(1 - t\cos t).$$

Question) Now what relation do we have between solutions?

The Nonhomogeneous Case

Even though the Linearity Principle does not hold for a non-homogeneous linear equation, we have (Detail 2)

Extended Linearity Principle

Consider

$$\frac{dy}{dt} = a(t)y + b(t)$$

and its associated homogeneous equation

$$\frac{dy}{dt} = a(t)y.$$

If y_h is any nonzero solution of the homogeneous equation and y_p is any solution of the nonhomogeneous equation, then $ky_h(t) + y_p(t)$ is the general solution of the nonhomogeneous equation.

Go back to the previous example

$$\frac{dy}{dt} = (\cos t)y + \frac{1}{5}(1 - t\cos t).$$

General solution:

$$y(t) = \frac{t}{5} + ke^{\sin t}, \quad \text{(Detail 3)}$$

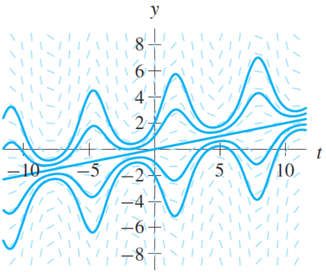


Figure 1.94
The slope field and graphs of various solutions to

$$\frac{dy}{dt} = (\cos t)y + \tfrac{1}{5}(1 - t\cos t).$$

To solve a linear equation,

1. find the general solution to its associated homogeneous equation, separating variables,
2. find a particular solution of the nonhomogeneous equation,
3. add them to get the general solution of the nonhomogeneous equation.

Which part is most difficult in practice?
([PRG], p.116)

The Lucky Guess

e.g.) Consider

$$\frac{dy}{dt} = -2y + e^t.$$

1. The general solution to its associated homogeneous equation $dy/dt = -2y$ is ke^{-2t}
2. A particular solution to the nonhomogeneous equation is $\frac{1}{3}e^t$ (Detail 4),
3. The general solution to the nonhomogeneous equation is

$$y(t) = ke^{-2t} + \frac{1}{3}e^t.$$

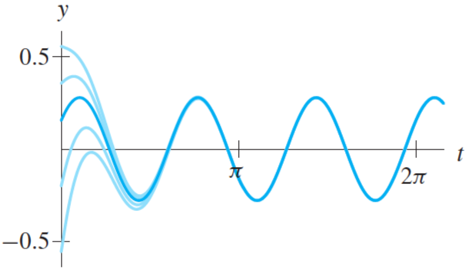
Another Lucky Guess

e.g) Consider

$$\frac{dy}{dt} + 2y = \cos 3t. \quad \text{(Detail 5)}$$

The general solution:

$$y(t) = ke^{-2t} + \frac{2}{13}\cos 3t + \frac{3}{13}\sin 3t.$$



Qualitative Analysis

Consider

$$\frac{dy}{dt} = \lambda y + b(t)$$

for negative λ.
The general solution:

$$y(t) = ke^{\lambda t} + y_p(t)$$

where y_p is a particular solution.
All solutions are close to $y_p(t)$ for large t.
([PRG], p. 119)

- Sometimes, our first guess may not work. What do we have to do? Guess again.
- Consider
$$\frac{dy}{dt} = -2y + 3e^{-2t}.$$
- The general solution to the homogeneous equation: $y(t) = ke^{-2t}$.
- Guessing $y_p(t) = \alpha e^{-2t}$ does not work (Detail 6)
- Second guess: $y_p(t) = \alpha te^{-2t}$. (Detail 7)
- The general solution to the nonhomogeneous equation:
$$y(t) = ke^{-2t} + 3te^{-2t}.$$

([PRG], p. 119)

Overview

1. Sect. 1.8 Linear Equations
   - Linear Differential Equations
   - Linearity Principles
   - Solving Linear Equations
   - Qualitative Analysis
   - Second Guessing
   - homework

What's next: Sect. 1.9 Integrating Factors for Linear Equations

homework

- Suggested Exercises (optional): 1-5 odd, 7-11 odd, 13, 17, 20, 21, , 29, 33.
- Homework Exercises (required to submit): 1-5 odd, 7- 11 odd, 13, 29
- When it comes to quiz, it is enough to study homework exercises whereas in order to prepare for exam, it would be better to solve the suggested exercises.
- For Exercise 1-5 odd, 7-11 odd, you have to explain how you come up with your guessing based on the Method of Undetermined Coefficient.

References

Paul Blanchard, Robert L. Devaney, Glen R. Hall
Differential Equations, fourth edition.