Chapter 1 First Order Differential Equations

Sect. 1.9 Integrating Factors for Linear Equations

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Overview

Example

- Consider

$$\frac{dy}{dt} + \frac{2}{t}y = t - 1.$$

- What if we multiply by t^2?

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$$\frac{dy}{dt} + \frac{2}{t}y = t - 1.$$
- What if we multiply by t^2?
- Question) How can we come up with t^2?

Integrating Factors

- Consider

$$\frac{dy}{dt} + g(t)y = b(t). \quad \text{(Detail 1)}$$

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- Assume $\mu(t)$ is a function satisfying $\frac{d\mu}{dt} = \mu(t)g(t)$.
- Multiplying the given equation by μ,

$$\frac{d(\mu(t)y(t))}{dt} = \mu(t)b(t).$$

- Consequently

$$y(t) = \frac{1}{\mu(t)} \int \mu(t)b(t)dt.$$

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$$\mu(t) = e^{\int g(t)dt}$$

The function $\mu(t)$ is called an **integrating factor**.

Complete Success

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Multiplying by $\mu(t)$

$$\frac{d}{dt}(t^2 y) = t^2(t-1).$$

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Integrating

$$y(t) = \frac{t^2}{4} - \frac{t}{3} + \frac{k}{t^2}.$$

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$$y(t) = \frac{t^2}{4} - \frac{t}{3} + \frac{k}{t^2}.$$

It is also a good illustration of the Extended Linearity Principle. (Detail 4)

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Integrating factor:

$$\mu(t) = e^{\int -t^2 dt} = e^{-t^3/3}$$

Multiplying by μ

$$\frac{d}{dt}\left(e^{-t^3/3} y\right) = e^{-t^3/3}(t-1)$$

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Then we are stuck.

Overview of Sect. 1.9

1. 1.9 Integrating Factors for Linear Equations
   - Integrating Factors
   - homework
2. Overview of Chapter 1

What's next: Chapter 2 First-Order Systems

homework

- Suggested Exercises (optional): 1-5 odd, 7-11 odd, 13-17 odd, 19, 21
- Homework Exercises (required to submit): 1-5 odd, 7, 9, 13, 21

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3. Qualitative Technique: Slope Fields
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7. Bifurcations
8. Linear Equations
9. Integrating Factors for Linear Equations

References

Paul Blanchard, Robert L. Devaney, Glen R. Hall
Differential Equations, fourth edition.