Chapter 2 First-Order Systems

Sect. 2.2 The Geometry of Systems

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Overview

The Predator-Prey Vector Field

Consider

$$\frac{dP}{dt} = 2R - 1.2RF$$
$$\frac{dF}{dt} = -F + 0.9RF$$

Let

$$\boldsymbol{P}(t) = \begin{pmatrix} R(t) \\ F(t) \end{pmatrix}, \quad \frac{d\boldsymbol{P}}{dt} = \begin{pmatrix} dR/dt \\ dF/dt \end{pmatrix}$$

Then the system can be rewritten as

$$\frac{d\boldsymbol{P}}{dt} = \begin{pmatrix} 2R - 1.2RF \\ -F + 0.9RF \end{pmatrix}.$$

The Predator-Prey Vector Field

- Continue to consider

$$\frac{d\boldsymbol{P}}{dt} = \begin{pmatrix} 2R - 1.2RF \\ -F + 0.9RF \end{pmatrix}.$$

- Now RHS is a function that assigns a vector to a vector.
- Denote RHS by $\boldsymbol{F}$

$$\boldsymbol{F}(\boldsymbol{P}) = \begin{pmatrix} 2R - 1.2RF \\ -F + 0.9RF \end{pmatrix}.$$

- Then the system turns into

$$\frac{d\boldsymbol{P}}{dt} = \boldsymbol{F}(\boldsymbol{P}).$$

The Vector Field for A Simple Harmonic Oscillator

Consider the harmonic oscillator for $k/m = 1$

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = -y.$$

Denote *RHS* by $F(y, v) = (v, -y)$. (Detail 1)

([PRG], p. 168)

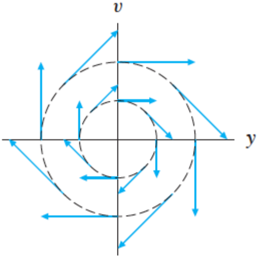


Figure 2.18
Selected vectors in the vector field $\mathbf{F}(y, v) = (v, -y)$

The Vector Field for a Simple Harmonic Oscillator

To avoid the confusion of overlapping vectors, we scale the vectors so they all have the same length. The resulting picture is called the **direction field**.

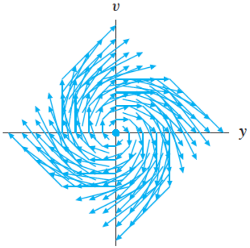


Figure 2.19
Vector field for $\mathbf{F}(y, v) = (v, -y)$.

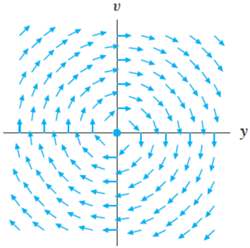


Figure 2.20
Direction field for $\mathbf{F}(y, v) = (v, -y)$.

Question) Can we get some picture of solutions based on direction field as we did for slope field?

The Geometry of Solutions

We can use a direction field to sketch solution curves.

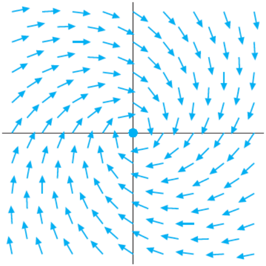


Figure 2.24
A direction field that spirals about the origin.

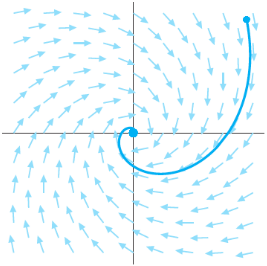


Figure 2.25
A solution curve corresponding to the initial condition indicated.

([PRG], p.172)

The Geometry of Solutions

A solution curve for the harmonic oscillator:

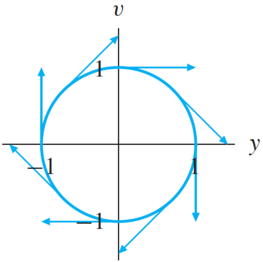


Figure 2.26

A solution curve to a predator-prey system:

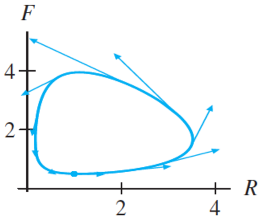


Figure 2.27

Example) A Population Model for Two Competing Species

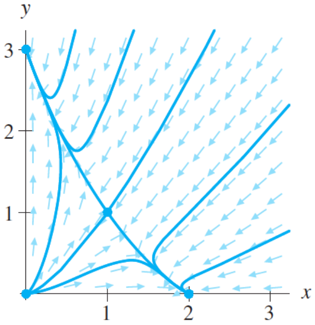
Consider

$$\frac{dx}{dt} = 2x\left(1 - \frac{x}{2}\right) - xy$$
$$\frac{dy}{dt} = 3y\left(1 - \frac{y}{3}\right) - 2xy.$$

Equilibrium points (Detail 2): (0,0), (0,3), (2,0), (1,1)
Question) Can we understand the system geometrically? ([PRG], p. 175)

Example) A Population Model for Two Competing Species

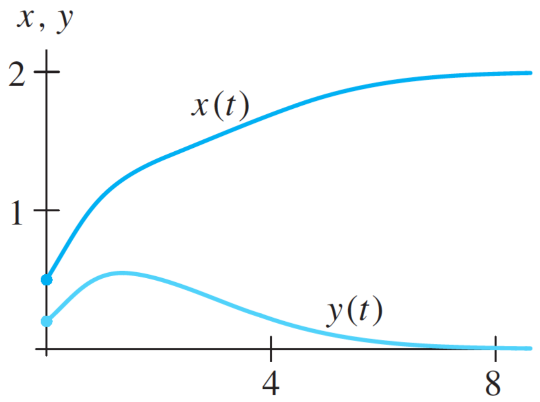
We can sketch solution curves



Question) For a solution curve converging to (2,0), when does y become extinct?

Example) A Population Model for Two Competing Species

To answer the question, we have to draw $x(t)$- and $y(t)$-graphs



Qualitative Thinking

Recall how we draw a solution curve based on the portrait

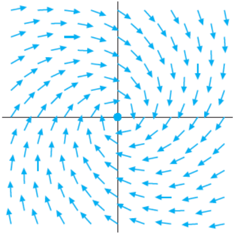


Figure 2.24

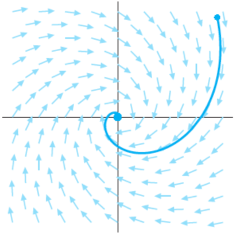


Figure 2.25

Question: What if the phase portrait varies as time goes by?

- It will be discussed in Chapt. 4 Forcing and Resonance.
- Throughout Chapter 2, we only consider **autonomous** systems.

([PRG], p. 177)

Overview

1. 2.2 The Geometry of Systems
   - The Predator-Prey Vector Field
   - The Vector Field for A Simple Harmonic Oscillator
   - The Geometry of Solutions
   - Example) A Population Model for Two Competing Species
   - Qualitative Thinking
   - Homework

What's next: 2.3 The Damped Harmonic Oscillator (Analytic Technique)

Homework

- Suggested Exercises (optional): 9, 11, 13(a), 15 (a), 21, 23, 25,
- Homework Exercises (required to submit): 11, 13(a), 15(a), 21, 25.

References

Paul Blanchard, Robert L. Devaney, Glen R. Hall
Differential Equations, fourth edition.