Chapter 2 First-Order Systems

Sect. 2.6 Existence and Uniqueness for Systems

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Overview

Example

As opposed to differential equations, for systems, solutions might "intersect" themselves.

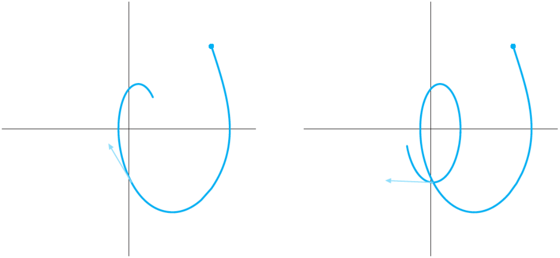

Figure 2.55

We don't have the Uniqueness Theorem for systems? What is happening to this example?

The Existence and Uniqueness Theorem

Actually, we have

The Existence and Uniqueness Theorem

Let
$$\frac{d}{dt}\boldsymbol{Y} = \boldsymbol{F}(t, \boldsymbol{Y})$$
be a system of DE. Suppose t_0 is an initial time and Y_0 is an initial value. Suppose $\boldsymbol{F}$ is continuously differentiable. Then there exists an $\varepsilon > 0$ and $\boldsymbol{Y}(t)$ defined for $t_0 - \varepsilon < t < t_0 + \varepsilon$ such that $\boldsymbol{Y}(t)$ satisfies the initial- value problem
$$\frac{d}{dt}\boldsymbol{Y} = \boldsymbol{F}(t, \boldsymbol{Y}), \quad \boldsymbol{Y}(t_0) = Y_0.$$
Moreover, for t in the this interval, this solution is unique.

([PRG],p.204)

Consequences of Uniqueness for Autonomous Systems

Roughly speaking, the Uniqueness Theorem says

> *two solutions cannot start at the same place at the same time.*

For autonomous systems, we have two consequences.

1) The solution curve for a single solution cannot loop back and intersect itself unless the solution is periodic and the solution curve is a simple, closed curve.

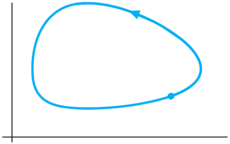

Figure 2.54

([PRG],p.205)

Consequences of Uniqueness for Autonomous Systems

But nonautonomous systems do not have the property

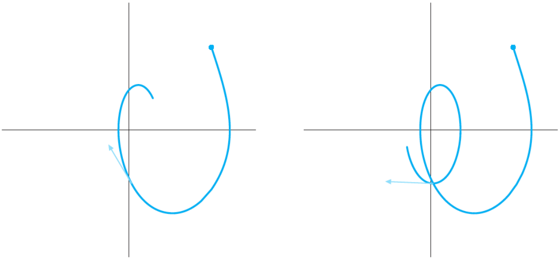

Figure 2.55

because the solution curve might intersect itself at <u>different times</u>.

Consequences of Uniqueness for Autonomous Systems

2) the solution curves for two different solutions cannot intersect unless they sweep out the same curve.

e.g.) Consider

$$\frac{d^2y}{dt^2} + y = 0.$$

(Detail 1)

Short Review

For autonomous systems, we have two consequences of uniqueness.

1) The solution curve for a single solution cannot loop back and intersect itself unless the solution is periodic and the solution curve is a simple, closed curve.
2) The solution curves for two different solutions cannot intersect unless they sweep out the same curve.

Consequences for Two-Dimensional Autonomous Systems

Consider an autonomous system that involves only two dependent variables (two-dimensional autonomous systems).

1. If a solution curve is a closed curve in the phase plane, then any solution with an initial condition that is inside the curve is trapped for all time.
2. A number of solution curves can form a "fence".

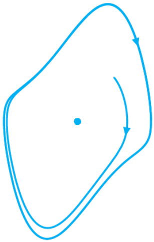

Figure 2.56

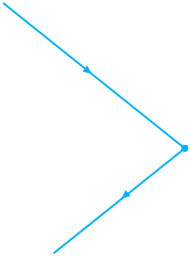

Figure 2.57

Formal Verification of the Consequences for Autonomous Systems

1. Let $\boldsymbol{Y}_1$, $\boldsymbol{Y}_2$ be solutions to an autonomous system $\frac{d}{dt}\boldsymbol{Y} = \boldsymbol{F}(\boldsymbol{Y})$.
2. Suppose they intersect at a point $\boldsymbol{Y}_0$ in the phase plane. That is,
$$\boldsymbol{Y}_1(t_1) = \boldsymbol{Y}_0 = \boldsymbol{Y}_2(t_2)$$
3. Consider
$$\boldsymbol{Y}_3(t) = \boldsymbol{Y}_1(t - (t_2 - t_1)).$$
4. Note $\boldsymbol{Y}_3(t)$ is a solution and $\boldsymbol{Y}_3(t_2) = \boldsymbol{Y}_1(t_1)$. (Detail 2).
5. Therefore $Y_2(t) = Y_3(t)$ for all t. In other words
$$\boldsymbol{Y}_2(t) = \boldsymbol{Y}_1(t - (t_2 - t_1)).$$

Exercise 8

(a) Suppose $\boldsymbol{Y}_1(t)$ is a solution of an autonomous system $d\boldsymbol{Y}/dt = \boldsymbol{F}(\boldsymbol{Y})$. For any constant t_0, is $\boldsymbol{Y}_2(t) = \boldsymbol{Y}_1(t + t_0)$ a solution ? (Detail 3)

(b) What is the relationship between the solution curves of $\boldsymbol{Y}_1(t)$ and $\boldsymbol{Y}_2(t)$.

Exercise 9

Suppose $\boldsymbol{Y}_1(t)$ and $\boldsymbol{Y}_2(t)$ are solutions of an autonomous system $d\boldsymbol{Y}/dt = \boldsymbol{F}(\boldsymbol{Y})$, where $\boldsymbol{F}(\boldsymbol{Y})$ satisfies the hypotheses of the Uniqueness Theorem. Suppose also that $\boldsymbol{Y}_2(1) = \boldsymbol{Y}_1(0)$. How are $\boldsymbol{Y}_1(t)$ and $\boldsymbol{Y}_2(t)$ related?

Overview of Sect. 2.6

What's next: Chapt. 3 Linear Systems

Overview of Chapt. 2

Homework

- Suggested Exercises (optional): 1(a), 3, 9, 11
- Homework Exercises (required to submit): 1(a), 3, 9
- Exercise 11 is highly recommended to solve. The only reason I did not assign it as a homework exercise is because this exercise might be difficult.

References

Paul Blanchard, Robert L. Devaney, Glen R. Hall
Differential Equations, fourth edition.