Chapter 3 Linear Systems

Sect. 3.1 Properties of Linear Systems and The Linearity Principle

Jeaheang(Jay) Bang

Rutgers University

j.bang@rutgers.edu

Thu. July 20, 2017

Overview of Chapt. 3 Linear Systems

1. Properties of Linear Systems and the Linearity Principle
2. Staright-Line Solutions
3. Phase Portraits for Linear Systems with Real Eigenvalues
4. Complex Eigenvalues
5. Special Cases: Repeated and Zero Eigenvalues
6. Second-Order Linear Equations
7. The Trace-Determinant Plane
8. Linear Systems in Three Dimensions.

Overview of Chapt. 3 Linear Systems

In Chapt. 3,

- we focus on autonomous linear systems,
- we show how to use the algebraic and geometric forms of the vector field to produce the general solution of an autonomous linear system,
- the qualitative behavior of linear systems leads to a classification scheme for these systems.
- we continue to study the damped harmonic oscillator.

Overview of Sect. 3.1

The Harmonic Oscillator

The harmonic oscillator:

$$m\frac{d^2y}{dt} + b\frac{dy}{dt} + ky = 0.$$

Letting $v = dy/dt$,

$$\begin{aligned}\frac{dy}{dt} &= v \\ \frac{dv}{dt} &= -\frac{k}{m}y - \frac{b}{m}v.\end{aligned}$$

([PRG], p. 240)

Two Cafés

Let

$$x(t) = \text{daily profit of Paul's café at time t}$$
$$y(t) = \text{daily profit of Bob's café at time t.}$$

The system is

$$\frac{dx}{dt} = ax + by$$
$$\frac{dy}{dt} = cx + dy,$$

where a, b, c, d are parameters.
([PRG], p. 241)

Two Cafés

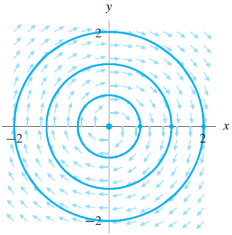

Figure 3.1
The direction field and three solution curves for the system

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -x.$$

Note that all three curves are circles centered at the origin.

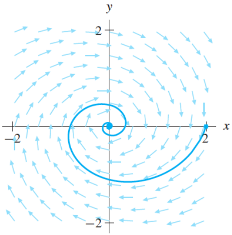

Figure 3.2
The direction field and a solution curve for the system

$$\frac{dx}{dt} = -x + 4y$$
$$\frac{dy}{dt} = -3x - y.$$

This solution curve spirals toward the origin as t increases.

Two Cafés

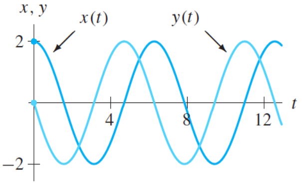

Figure 3.3
The $x(t)$- and $y(t)$-graphs corresponding to the solution curve in Figure 3.1, with initial condition $(x_0, y_0) = (2, 0)$.

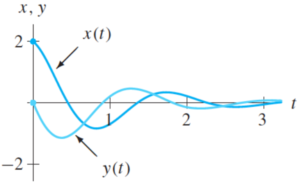

Figure 3.4
The $x(t)$- and $y(t)$-graphs corresponding to the solution curve in Figure 3.2 with initial condition $(x_0, y_0) = (2, 0)$.

Linear Systems and Matrix Notation

- In this chapter, we mainly consider

$$\frac{dx}{dt} = ax + by$$
$$\frac{dy}{dt} = cx + dy$$

  where a, b, c and d are constants.
- Such a system is said to be a **linear system with constant coefficients**.
- The constants a, b, c, d are the **coefficients**.
- These systems are also called **planar (or two-dimensional) linear systems**.

([PRG], p. 243)

Linear Systems and Matrix Notation

Consider

$$\frac{dx}{dt} = ax + by$$
$$\frac{dy}{dt} = cx + dy$$

. Let

$$\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \boldsymbol{Y} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Then the system turns into

$$\frac{d\boldsymbol{Y}}{dt} = \boldsymbol{AY}.$$

Equilibrium Points for Linear Systems and the Determinant

Find the equilibrium solutions. Set $\boldsymbol{AY} = 0$. That is,

$$\begin{aligned} ax + by &= 0 \\ cx + dy &= 0. \end{aligned}$$

Any constants x, y satisfying the above equations are equilibrium solutions.
([PRG], p. 246)

Equilibrium Points for Linear Systems and the Determinant

Theorem

If $\boldsymbol{A}$ is a matrix with $\det \boldsymbol{A} \neq 0$, then the only equilibrium point of the linear system $d\boldsymbol{Y}/dt = \boldsymbol{A}\boldsymbol{Y}$ is the origin.

- If $\det \boldsymbol{A} = 0$, it is called **singular** or **degenerate**.
- Otherwise it is called **nondegenerate**.

The Linearity Principle

Linearity Principle

Suppose $d\mathbf{Y}/dt = \mathbf{A}\mathbf{Y}$ is a linear system of DE

1. If $\mathbf{Y}(t)$ is a solution of this system and k is any constant, then $k\mathbf{Y}(t)$ is also a solution,
2. If $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ are two solutions of this system, then $\mathbf{Y}_1(t) + \mathbf{Y}_2(t)$ is also a solution. So
3. if $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ are two solutions of this system, then

$$k_1\mathbf{Y}_1(t) + k_2\mathbf{Y}_2(t)$$

is a solution.

Why? (Detail 1)

The Linearity Principle

- Consider

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 2 & 3 \\ 0 & -4 \end{pmatrix}$$

- We found that

$$\mathbf{Y}_1(t) = \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix}, \quad \mathbf{Y}_2(t) = \begin{pmatrix} -e^{-4t} \\ 2e^{-4t} \end{pmatrix}$$

  are solutions.
- Based on the Linearity Principle, any linear combination of these solutions is again a solution.
- How about the geometry?

The Linearity Principle

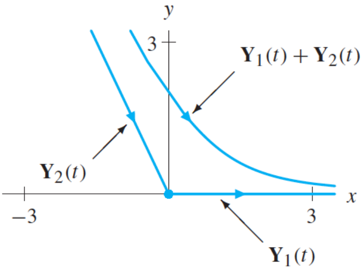

Figure 3.5
The Linearity Principle implies that the function $\mathbf{Y}_1(t) + \mathbf{Y}_2(t)$ is a solution of the system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 2 & 3 \\ 0 & -4 \end{pmatrix} \mathbf{Y}$$

because it is the sum of the two solutions $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$.

([PRG], p. 249)

Initial-Value Problem and the General Solution

So far, considering

$$\frac{d\boldsymbol{Y}}{dt} = \begin{pmatrix} 2 & 3 \\ 0 & -4 \end{pmatrix} \boldsymbol{Y},$$

we found that $k_1 \boldsymbol{Y}_1(t) + k_2 \boldsymbol{Y}_2(t)$ is a solution where

$$\boldsymbol{Y}_1(t) = \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix}, \quad \boldsymbol{Y}_2(t) = \begin{pmatrix} -e^{-4t} \\ 2e^{-4t} \end{pmatrix}.$$

Question) Are they all?
([PRG], p. 255)

Initial-Value Problem and the General Solution

Yes they are all. Consider an initial-value problem

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 2 & 3 \\ 0 & -4 \end{pmatrix}, \quad \mathbf{Y}(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

For the solution $\mathbf{Y}$, we can write it as a linear combination:

$$\mathbf{Y}(t) = k_1 \mathbf{Y}_1(t) + k_2 \mathbf{Y}_2(t).$$

(Detail 2)

Initial-Value Problem and the General Solution

Theorem

- Suppose $\boldsymbol{Y}_1(t)$ and $\boldsymbol{Y}_2(t)$ are solutions of the linear system $d\boldsymbol{Y}/dt = \boldsymbol{AY}$.
- If $\boldsymbol{Y}_1(0)$ and $\boldsymbol{Y}_2(0)$ are linearly independent,
- then for any initial condition $\boldsymbol{Y}(0) = (x_0, y_0)$,
- we can find constants k_1, k_2 so that $k_1\boldsymbol{Y}_1 + k_2\boldsymbol{Y}_2$ is the solution to

$$\frac{d\boldsymbol{Y}}{dt} = \boldsymbol{AY}, \quad \boldsymbol{Y}(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

.

Initial-Value Problem and the General Solution

Take-home lesson)
Now in order to find all the solutions to a linear system, we only need to find two particular solutions with linearly independent initial positions.

An Undamped Harmonic Oscillator

- Consider

$$\frac{d^2y}{dt^2} = -y.$$

- Know $y_1(t) = \cos t, y_2(t) = \sin t$.
- By Linearity Principle,

$$\mathbf{Y}(t) = k_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + k_2 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

  is the general solution. (Detail 3)
- Therefore,

$$y(t) = k_1 \cos t + k_2 \sin t.$$

Question) Do we have to rely on systems? No we don't have to. It will be discussed in Sec. 3.6 Second-Order Linear DE.
([PRG], p. 256)

Overview of Sect. 2.6

1. Sect. 3.1 Properties of Linear Systems and The Linearity Principle
   - The Harmonic Oscillator and Two Cafés
   - Linear Systems and Matrix Notation
   - Equilibrium Points for Linear Systems and the Determinant
   - The Linearity Principle
   - Initial-Value Problem and the General Solution
   - An Undamped Harmonic Oscillator
   - Homework

What's next: Chapt. 3.2. Straight-Line Solutions

Homework

- Suggested Exercises (optional): 15, 17, 19, 25, 27, 31, 35
- Homework Exercises (required to submit): 15, 17, 27

References

Paul Blanchard, Robert L. Devaney, Glen R. Hall
Differential Equations, fourth edition.