Chapter 3 Linear Systems

Sect. 3.2 Straight-Line Solutions

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Overview

Geometry of Straight-Line Solutions

Main idea: we use the geometry of the vector field to find special solutions of linear systems.

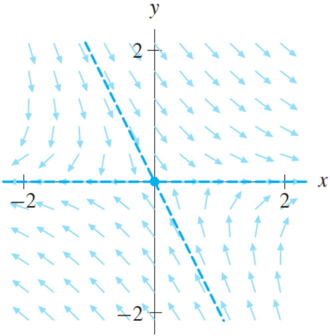

Figure 3.7
The direction field for the linear system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 2 & 3 \\ 0 & -4 \end{pmatrix} \mathbf{Y}.$$

There are two special lines through the origin. On the x-axis, the vectors in the direction field all point directly away from the origin. On the distinguished line that runs from the second quadrant to the fourth quadrant, all vectors of the direction field point directly toward the origin.

([PRG], p.264)

Geometry of Straight-Line Solutions

- In Section 3.1, we saw that

$$\mathbf{Y}_1(t) = \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix}, \quad \mathbf{Y}_2(t) = \begin{pmatrix} -e^{-4t} \\ 2e^{-4t} \end{pmatrix}.$$

  are two linearly independent solutions for the system $d\mathbf{Y}/dt = \mathbf{A}\mathbf{Y}$.
- We can consider the geometry of these solutions in the phase plane. (Detail 1)
- Tehy confirm what we guessed by looking at the direction field.

Geometry of Straight-Line Solutions

Consider $d\mathbf{Y}/dt = \mathbf{AY}$.

- Goal: Find a stright-line solution.
- If $\mathbf{V} = (x, y)$ is a straight-line solution, then the vector field at (x, y) must point either in the same direction or in exactly the opposite direction as the vector from $(0, 0)$ to (x, y).
- So
$$\mathbf{AV} = \lambda \mathbf{V}$$
for some scalar λ.
- So
$$\frac{d\mathbf{V}}{dt} = \lambda \mathbf{V}.$$

Theorem

- Suppose $\boldsymbol{A}$ has a real eigenvalue λ with eigenvector $\boldsymbol{V}$.
- Then $d\boldsymbol{Y}/dt = \boldsymbol{AY}$ has the straight-line solution

$$\boldsymbol{Y}(t) = e^{\lambda t}\boldsymbol{V}.$$

- Moreover, if λ_1, λ_2 are distinct, real eigenvalues with eigenvectors $\boldsymbol{V}_1$, $\boldsymbol{V}_2$ respectively,
- then $\boldsymbol{Y}_1(t) = e^{\lambda_1 t}\boldsymbol{V}_1$, $\boldsymbol{Y}_2(t) = e^{\lambda_2 t}\boldsymbol{V}_2$ are linearly independent and

$$\boldsymbol{Y}(t) = k_1 e^{\lambda_1 t}\boldsymbol{V}_1 + k_2 e^{\lambda_2 t}\boldsymbol{V}_2$$

is the general solution of the system.

Putting Everything Together

- Consider

$$\frac{d\boldsymbol{Y}}{dt} = \boldsymbol{BY} = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} \boldsymbol{Y}.$$

- Eigenvalues of $\boldsymbol{B}$: $\lambda_1 = 4, \lambda_2 = 1$ (Detail 2)
- Eigenvectors: $\boldsymbol{V}_1 = (1, 1), \boldsymbol{V}_2 = (-2, 1)$. (Detail 3)
- The general solution:

$$\boldsymbol{Y}(t) = k_1 e^{4t} \boldsymbol{V}_1 + k_2 e^{t} \boldsymbol{V}_2.$$

([PRG], p.275)

Putting Everything Together

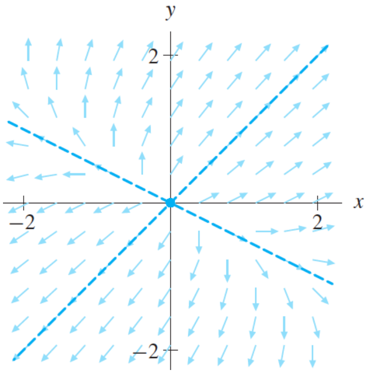

Figure 3.10

The direction field for the system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{B}\mathbf{Y} = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} \mathbf{Y}.$$

Note the two distinguished lines of eigenvectors. The one in the first quadrant corresponds to the solution $\mathbf{Y}_1(t) = e^{4t}(1, 1)$ and the one in the second quadrant corresponds to the solution $\mathbf{Y}_2(t) = e^{t}(-2, 1)$.

A Harmonic Oscillator

- Consider
$$\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 10y = 0.$$
- The corresponding system:
$$\frac{d\mathbf{Y}}{dt} = \mathbf{C}\mathbf{Y}, \quad \text{where } \mathbf{C} = \begin{pmatrix} 0 & 1 \\ -10 & -7 \end{pmatrix} \text{ and } \mathbf{Y} = \begin{pmatrix} y \\ v \end{pmatrix}$$
(Detail 4)
- Eigenvalues: $\lambda_1 = -5, \lambda_2 = -2$. (Detail 5)
- Eigenvectors: $\mathbf{V}_1 = (1, -5), \mathbf{V}_2 = (1, -2)$. (Detail 6)
- The general solution to the system:
$\mathbf{Y}(t) = k_1 e^{-5t}\mathbf{V}_1 + k_2 e^{-2t}\mathbf{V}_2$.
- The general solution to the 2nd-order DE:
$y(t) = k_1 e^{-5t} + k_2 e^{-2t}$

A Harmonic Oscillator

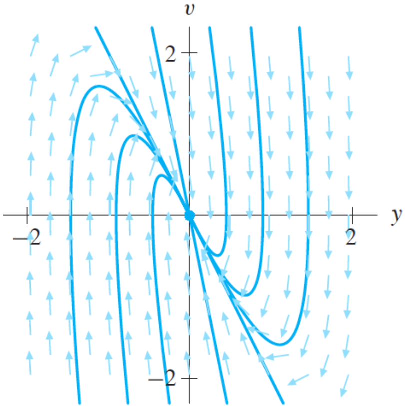

Figure 3.11
Phase portrait for

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -10 & -7 \end{pmatrix} \mathbf{Y}.$$

This linear system is obtained from the harmonic oscillator

$$\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 10y = 0,$$

where $\mathbf{Y} = (y, v)$ and $v = dy/dt$.

([PRG], p.275)

Overview

What's next: Chapt. 3.3 Phase Portraits for Linear Systems with Real Eigenvalues

Homework

- Suggested Exercises (optional): 1-5 odd except (c), 11, 13, 19, 21-23 odd except (d)
- Homework Exercises (required to submit): 1-3 odd except (c), 11, 13,

References

Paul Blanchard, Robert L. Devaney, Glen R. Hall
Differential Equations, fourth edition.