Chapter 3 Linear Systems

Sect. 3.3 Phase Portraits for Linear Systems with Real Eigenvalues

Jeaheang(Jay) Bang

Rutgers University

j.bang@rutgers.edu

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Overview

Saddles

- In this section, we want to use the behavior of straight-line solutions to determine the behavior of all solutions.
- Let us first consider

$$\frac{d\boldsymbol{Y}}{dt} = \boldsymbol{AY} = \begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix} \boldsymbol{Y}.$$

(Detail 1)

Saddles

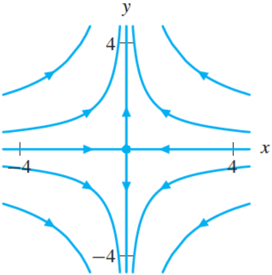

Figure 3.12
Phase portrait for the system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY} = \begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{Y}.$$

This type of equilibrium point (one positive and one negative eigenvalue) is called a **saddle**.
([PRG], p.280)

Saddles

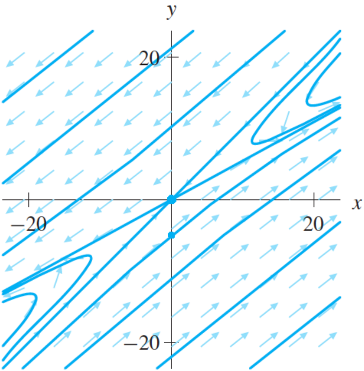

Figure 3.14

The direction field and phase portrait for the system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{B}\mathbf{Y} = \begin{pmatrix} 8 & -11 \\ 6 & -9 \end{pmatrix} \mathbf{Y}.$$

The eigenvectors lie along the two distinguished lines that run through the first and third quadrants. Although some of the other solution curves look almost straight, they really curve slightly.

Sinks

Consider

$$\frac{d\mathbf{Y}}{dt} = \mathbf{C}\mathbf{Y} = \begin{pmatrix} -1 & 0 \\ 0 & -4 \end{pmatrix} \mathbf{Y}$$

(Detail 2)

Sinks

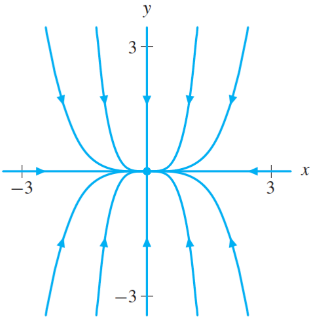

Figure 3.16
The phase portrait for the system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{C}\mathbf{Y} = \begin{pmatrix} -1 & 0 \\ 0 & -4 \end{pmatrix} \mathbf{Y}.$$

Note that all solution curves tend to the equilibrium point at the origin.

This type of equilibrium point (two negative eigenvalues) is called a **sink**. ([PRG], p.285)

Sinks

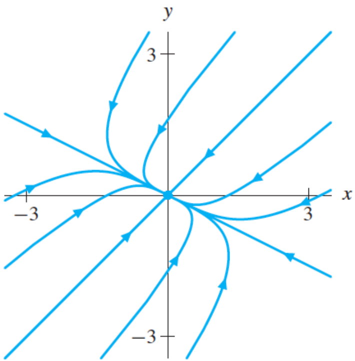

Figure 3.17
Phase portrait for the system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{DY} = \begin{pmatrix} -2 & -2 \\ -1 & -3 \end{pmatrix} \mathbf{Y}.$$

All solutions tend to the equilibrium point at the origin, and all solutions with the exception of the straight-line solutions associated to $\lambda_1 = -4$ tend to the origin tangent to the line of eigenvectors for $\lambda_2 = -1$.

Sources

Consider

$$\frac{d\boldsymbol{Y}}{dt} = \boldsymbol{E}\boldsymbol{Y} = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} \boldsymbol{Y}.$$

(Detail 3)

Sources

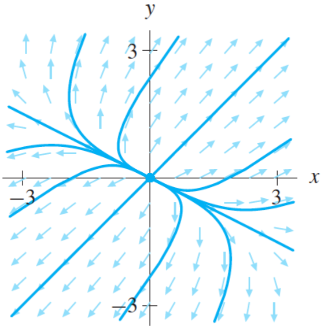

Figure 3.19
Phase portrait for the system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{E}\mathbf{Y} = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} \mathbf{Y}.$$

Note that, since $\mathbf{E} = -\mathbf{D}$, we can obtain the phase portrait for this example from the phase portrait for $d\mathbf{Y}/dt = \mathbf{D}\mathbf{Y}$. The solution curves are identical, but solutions travel away from the origin as $t \to \infty$.

This type of equilibrium point (two positive eigenvalues) is called a **source**.
([PRG], p.289)

Stable and Unstable Equilibrium Points

Three types of Equilibrium Points

Consider a linear system with two nonzero, real, distinct eigenvalues λ_1, λ_2.

- If $\lambda_1 < 0 < \lambda_2$, then the origin is a saddle.
- If $\lambda_1 < \lambda_2 < 0$, then the origin is a sink.
- If $0 < \lambda_1 < \lambda_2$, then the origin is a source.

Stable and Unstable Equilibrium Points

- Sink is said to be **stable**.
- Saddle and source are said to be **unstable**.

([PRG], p.290)

Overview

1. Sect. 3.3 Phase Portraits for Linear Systems with Real Eigenvalues
    - Saddles
    - Sinks
    - Sources
    - Stable and Unstable Equilibrium Points
    - Homework

What's next: Chapt. 3.4 Complex Eigenvalues

Homework

- Suggested Exercises (optional): 1,5, 9,11, 19, 21, 27
- Homework Exercises (required to submit): 1, 5, 9, 19, 21

References

Paul Blanchard, Robert L. Devaney, Glen R. Hall
Differential Equations, fourth edition.