Chapter 3 Linear Systems

Sect. 3.4 Complex Eigenvalues

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Overview

1. Sect. 3.4 Complex Eigenvalues
   - A Linear System without Straight-Line Solutions
   - General Solutions for Systems with Complex Eigenvalues
   - Obtaining Real-Valued Solutions from Complex Solutions
   - The Qualitative Behavior of Systems with Complex Eigenvalues
   - A Spiral Source
   - Centers
   - Paul‘s and Bob‘s Cafés Revisited
   - Homework

A Linear System without Straight-Line Solutions
General Solutions for Systems with Complex Eigenvalues
Obtaining Real-Valued Solutions from Complex Solutions
The Qualitative Behavior of Systems with Complex Eigenvalues
A Spiral Source
Centers
Paul's and Bob's Cafés Revisited
Homework

A Linear System without Straight-Line Solutions

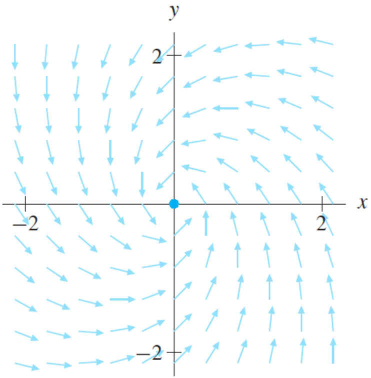

Figure 3.21
The direction field for

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -2 & -3 \\ 3 & -2 \end{pmatrix} \mathbf{Y}.$$

Apparently there are no straight-line solutions.

([PRG], p. 297)

General Solutions for Systems with Complex Eigenvalues

- Consider

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -2 & -3 \\ 3 & -2 \end{pmatrix} \mathbf{Y}$$

- Eigenvalues: $\lambda_1 = -2 + 3i, \lambda_2 = -2 - 3i$. (Detail 1)
- Question: How are we going to find solutions? What information do complex numbers give us?
- Strategy: Don't panic. Just keep going.

([PRG], p. 298)

General Solutions for Systems with Complex Eigenvalues

For $\lambda_1 = -2 + 3i$,

- an eigenvector: $\begin{pmatrix} i \\ 1 \end{pmatrix}$, (Detail 2)
- so $\boldsymbol{Y}(t) = e^{(-2+3i)t} \begin{pmatrix} i \\ 1 \end{pmatrix}$ (Detail 2) is a solution of the system.
- Use Euler's formula: $e^{it} = \cos t + i \sin t$,
- we can obtain

$$\boldsymbol{Y}(t) = \begin{pmatrix} -e^{-2t}\sin(3t) \\ e^{-2t}\cos(3t) \end{pmatrix} + i \begin{pmatrix} e^{-2t}\cos(3t) \\ e^{-2t}\sin(3t) \end{pmatrix}$$
$$= \boldsymbol{Y}_{re}(t) + i\,\boldsymbol{Y}_{im}(t)$$

.

General Solutions for Systems with Complex Eigenvalues

Then

$$\frac{d\mathbf{Y}_{re}}{dt} + i\frac{d\mathbf{Y}_{im}}{dt} = \frac{d}{dt}\mathbf{Y} = \mathbf{A}\mathbf{Y}_{re} + i(\mathbf{A}\mathbf{Y}_{im}).$$

So

$$\frac{d\mathbf{Y}_{re}}{dt} = \mathbf{A}\mathbf{Y}_{re}, \quad \frac{d\mathbf{Y}_{im}}{dt} = \mathbf{A}\mathbf{Y}_{im}.$$

That is, $\mathbf{Y}_{re}$, $\mathbf{Y}_{im}$ are solutions

Obtaining Real-Valued Solutions from Complex Solutions

Theorem

- Suppose $\boldsymbol{Y}(t)$ is a complex-valued solution to a linear system

$$\frac{d\boldsymbol{Y}}{dt} = \boldsymbol{A}\boldsymbol{Y} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \boldsymbol{Y}$$

  where a, b, c, d are all real.
- Suppose $\boldsymbol{Y}(t) = \boldsymbol{Y}_{re}(t) + i\,\boldsymbol{Y}_{im}(t)$ where $\boldsymbol{Y}_{re}, \boldsymbol{Y}_{im}$ are real-valued functions of t.
- Then $\boldsymbol{Y}_{re}, \boldsymbol{Y}_{im}$ are both solutions of the system $d\boldsymbol{Y}/dt = \boldsymbol{A}\boldsymbol{Y}$.

([PRG], p. 299)

Obtaining Real-Valued Solutions from Complex Solutions

- Going back to

$$\frac{d\boldsymbol{Y}}{dt} = \begin{pmatrix} -2 & -3 \\ 3 & -2 \end{pmatrix} \boldsymbol{Y},$$

- we have shown that

$$\boldsymbol{Y}_{re} = \begin{pmatrix} -e^{-2t}\sin(3t) \\ e^{-2t}\cos(3t) \end{pmatrix}, \quad \boldsymbol{Y}_{im} = \begin{pmatrix} e^{-2t}\cos(3t) \\ e^{-2t}\sin(3t) \end{pmatrix}$$

- By the Linearity Principle, the general solution:

$$k_1 \boldsymbol{Y}_{re} + k_2 \boldsymbol{Y}_{im}.$$

Obtaining Real-Valued Solutions from Complex Solutions

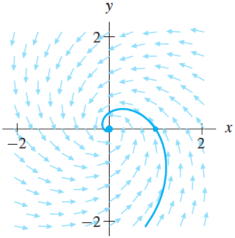

Figure 3.22
A solution curve in the phase plane for

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -2 & -3 \\ 3 & -2 \end{pmatrix} \mathbf{Y}.$$

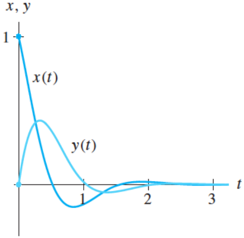

Figure 3.23
The $x(t)$- and $y(t)$-graphs of a solution to this differential equation.

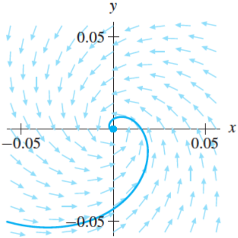

Figure 3.24
A magnification of Figure 3.22.

The Qualitative Behavior of Systems with Complex Eigenvalues

- Suppose $d\boldsymbol{Y}/dt = \boldsymbol{AY}$ is a linear system with $\lambda_1 = \alpha + i\beta, \lambda_2 = \alpha - i\beta,\ \beta \neq 0$.
- Then $\boldsymbol{Y}(t) = e^{(a+i\beta)t}\boldsymbol{Y}_0$ is a complex solution where $\boldsymbol{Y}_0$ is a (complex) eigenvector.
- If $\alpha > 0$, then $e^{\alpha t}$ term increases so the solution curve spirals off "toward infinity"
- If $\alpha < 0$, then $e^{\alpha t}$ term tends to zero so the solution curve tend to the origin
- If $\alpha = 0$, then $e^{\alpha t} = 1$ so the solutions oscillate and they are periodic.

([PRG], p. 303)

The Qualitative Behavior of Systems with Complex Eigenvalues

Linear Systems with Complex Eigenvalues

Given a linear system with complex eigenvalues $\lambda = \alpha \pm i\beta, \beta > 0$, the solution curves spiral around the origin in the phase plane with a period of $2\pi/\beta$. Moreover,

- If $\alpha < 0$, then the solutions spiral toward the origin. In this case, the origin is called a **spiral sink**.
- If $\alpha > 0$, then the solutions spiral away from the origin. In this case, the origin is called a **spiral source**.
- If $\alpha = 0$, then the solutions are periodic. In this case, the origin is called a **center**

A Spiral Source

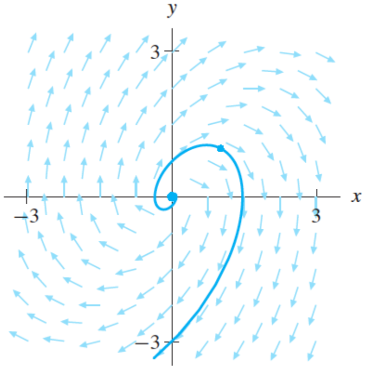

Figure 3.26

Direction field and the solution of the initial-value problem

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 2 \\ -3 & 2 \end{pmatrix} \mathbf{Y} \quad \text{and} \quad \mathbf{Y}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The solution spirals away from the origin.

([PRG], p. 306)

Centers

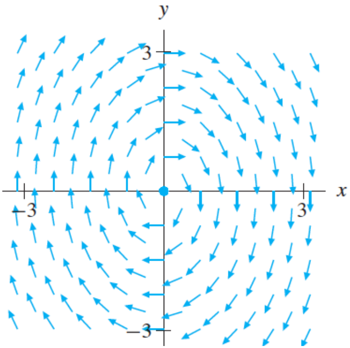

Figure 3.27
Direction field for the undamped harmonic oscillator system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{C}\mathbf{Y}, \quad \text{where } \mathbf{C} = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}.$$

Although the direction field suggests that the eigenvalues of the system are complex, we cannot determine by looking at the direction field if the origin is a center, a spiral source, or a spiral sink.

([PRG], p. 307)

Centers

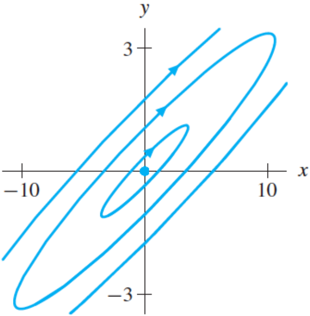

Figure 3.28
The phase portrait for the system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -3 & 10 \\ -1 & 3 \end{pmatrix} \mathbf{Y}.$$

All of the solution curves are ellipses, but their major and minor axes do not lie on the x- and y-axes.

Paul‘s and Bob‘s Cafés Revisited

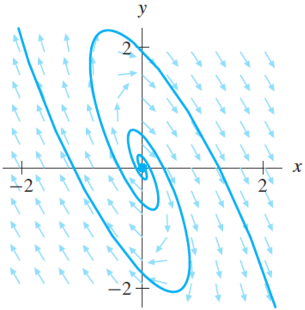

Figure 3.29
Phase portrait for

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 2 & 1 \\ -4 & -1 \end{pmatrix} \mathbf{Y}.$$

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What's next: Sect. 3. 5 Special Cases: Repeated and Zero Eigenvalues

Homework

- Suggested Exercises (optional): 3-5 odd except (e), 9-11 odd, 19, 23
- Homework Exercises (required to submit): 3-5 odd except (e), 9-11 odd

References

Paul Blanchard, Robert L. Devaney, Glen R. Hall
Differential Equations, fourth edition.