Chapter 3 Linear Systems

Sect. 3.5 Special Cases: Repeated and Zero Eigenvalues

Jeaheang(Jay) Bang

Rutgers University

j.bang@rutgers.edu

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Overview

A System with Repeated Eigenvalues

- Consider
$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y} = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix} \mathbf{Y}.$$
- Eigenvalue: $\lambda = -2$ (Detail 1)
- We only have one line of eigenvectors. (Detail 2)
- So, we only have one straight-line solution.
- We would like to attack the system differently (without using the Linearity Principle).

([PRG], p. 315)

A System with Repeated Eigenvalues

- The system can be re-written

$$\frac{dx}{dt} = -2x + y$$
$$\frac{dy}{dt} = -2y$$

- Because it is partially decoupled, (Detail 3)

$$y(t) = y_0 e^{-2t}$$
$$x(t) = y_0 t e^{-2t} + x_0 e^{-2t}.$$

- The general solution to the system:

$$\boldsymbol{Y}(t) = e^{-2t}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + te^{-2t}\begin{pmatrix} y_0 \\ 0 \end{pmatrix}$$

The Form of the General Solution

Consider

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}.$$

where $\mathbf{A}$ has a repeated eigenvalue λ. Then

$$\mathbf{Y}(t) = e^{\lambda t}\mathbf{V}_0 + te^{\lambda t}\mathbf{V}_1$$

is a solution if and only if (Detail 4)

$$\lambda \mathbf{V}_1 = \mathbf{A}\mathbf{V}_1, \quad \mathbf{V}_1 = (\mathbf{A} - \lambda \mathbf{I})\mathbf{V}_0.$$

([PRG], p. 318)

Theorem

- Consider $d\boldsymbol{Y}/dt = \boldsymbol{AY}$ where a 2×2 matrix $\boldsymbol{A}$ has a repeated real eigenvalue λ but only one line of eigenvectors.
- The the general solution has the form

$$\boldsymbol{Y}(t) = e^{\lambda t}\boldsymbol{V}_0 + te^{\lambda t}\boldsymbol{V}_1.$$

  where $\boldsymbol{V}_1$ is determined from $\boldsymbol{V}_0$ by

$$\boldsymbol{V}_1 = (\boldsymbol{A} - \lambda\boldsymbol{I})\boldsymbol{V}_0.$$

- If $\boldsymbol{V}_1 = 0$, then $\boldsymbol{V}_0$ is an eigenvector and $\boldsymbol{Y}(t)$ is a straight-line solution. Otherwise, $\boldsymbol{V}_1$ is an eigenvector.

The Form of the General Solution

Going back to the previous example, consider

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix} \mathbf{Y}.$$

1. the eigenvalue: $\lambda = -2$
2. Choose any initial condition $\mathbf{V}_0 = (x_0, y_0)$.
3. Then $\mathbf{V}_1 = (y_0, 0)$ (Detail 5)
4. According to the theorem, the general solution:

$$\mathbf{Y}(t) = e^{-2t} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + te^{-2t} \begin{pmatrix} y_0 \\ 0 \end{pmatrix}$$

Question) How can we draw phase portrait?

Qualitative Analysis of Systems with Repeated Eigenvalues

- Now in order to draw phase portrait, we factor out the e^{-2t} term

$$\boldsymbol{Y}(t) = e^{-2t}\left(\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + t\begin{pmatrix} y_0 \\ 0 \end{pmatrix}\right)$$

- Direction: $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + t\begin{pmatrix} y_0 \\ 0 \end{pmatrix}$

([PRG], p. 320)

Qualitative Analysis of Systems with Repeated Eigenvalues

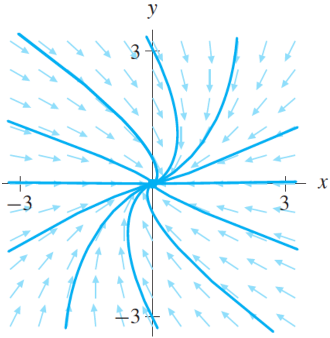

Figure 3.32
Phase portrait for the system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix} \mathbf{Y}.$$

Question) Can we generalize our qualitative analysis?

Qualitative Analysis of Systems with Repeated Eigenvalues

- In general consider $d\mathbf{Y}/dt = \mathbf{AY}$.
- The general solution:

$$\mathbf{Y}(t) = e^{\lambda t}\mathbf{V}_0 + te^{\lambda t}\mathbf{V}_1$$

  where λ is the eigenvalue and $\mathbf{V}_1 = (\mathbf{A} - \lambda I)\mathbf{V}_0$ is either an eigenvector or zero.
- Factor out the $e^{\lambda t}$ term;

$$\mathbf{Y}(t) = e^{\lambda t}(\mathbf{V}_0 + te^{\lambda t}\mathbf{V}_1).$$

- Thus the solution tends to the origin in a direction that is tangent to the line of eigenvectors.

Qualitative Analysis of Systems with Repeated Eigenvalues

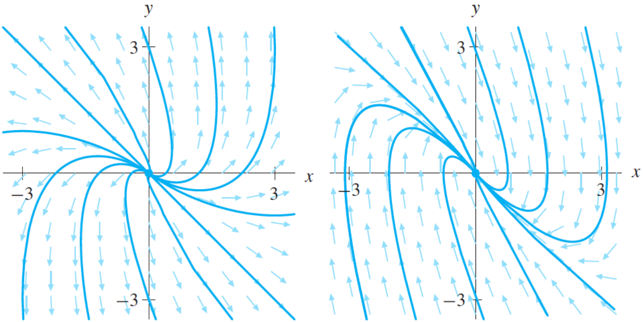

Figure 3.33
Typical phase portraits for systems with repeated eigenvalues.

A Harmonic Oscillator with Repeated Eigenvalues

- Consider

$$\frac{d^2y}{dt} + 2\sqrt{2}\frac{dy}{dt} + 2y = 0.$$

- Its associated system:

$$\frac{d\boldsymbol{Y}}{dt} = \boldsymbol{BY}, \quad \text{where } \boldsymbol{B} = \begin{pmatrix} 0 & 1 \\ -2 & -2\sqrt{2} \end{pmatrix}, \boldsymbol{Y} = \begin{pmatrix} y \\ v \end{pmatrix}$$

- Eigenvalue and eigenvector: $\lambda = -\sqrt{2}, \boldsymbol{V} = (1, -\sqrt{2})$.
- The general solution:

$$\boldsymbol{Y}(t) = e^{-\sqrt{2}t}\begin{pmatrix} y_0 \\ v_0 \end{pmatrix} + te^{-\sqrt{2}t}\begin{pmatrix} \sqrt{2}y_0 + v_0 \\ -2y_0 - \sqrt{2}v_0 \end{pmatrix}.$$

([PRG], p. 321)

A Harmonic Oscillator with Repeated Eigenvalues

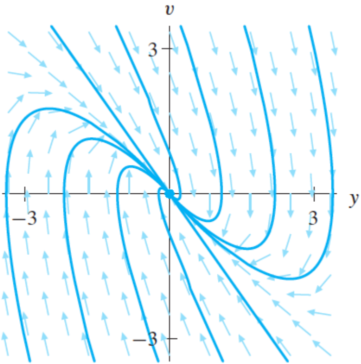

Figure 3.34
Direction field and solution curves for

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -2 & -2\sqrt{2} \end{pmatrix} \mathbf{Y}.$$

Note that the solution curves approach the origin tangent to the line

$$v = -\sqrt{2}\, y$$

of eigenvectors.

Paul's and Bob's Cafés One More Time

- Consider

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY} \begin{pmatrix} -5 & 1 \\ -1 & -3 \end{pmatrix} \mathbf{Y}.$$

- The matrix $\mathbf{A}$ has only one eigenvalue with one line of eigenvectors.

([PRG], p. 322)

Paul's and Bob's Cafés One More Time

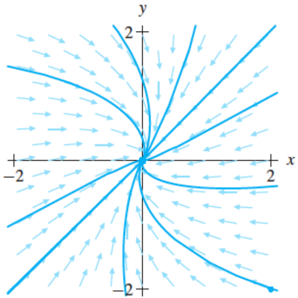

Figure 3.35
Phase portrait for the system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -5 & 1 \\ -1 & -3 \end{pmatrix} \mathbf{Y}.$$

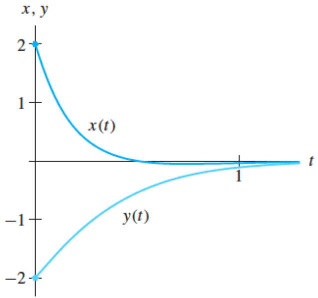

Figure 3.36
The $x(t)$- and $y(t)$-graphs for the solution with the initial condition indicated in Figure 3.35.

Question) What if we change the coefficients a little bit?

Paul‘s and Bob‘s Cafés One More Time

- Previously

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -5 & 1 \\ -1 & -3 \end{pmatrix} \mathbf{Y}.$$

- Now consider

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -5 & 1.1 \\ -1 & -3 \end{pmatrix} \mathbf{Y}.$$

  (Detail 6)

- Then the origin becomes a spiral sink.
- A little change in coefficient causes a distinct change in the qualitative behavior of the solutions, but...

Paul's and Bob's Cafés One More Time

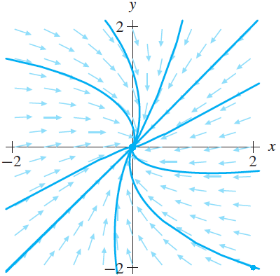

Figure 3.35
Before

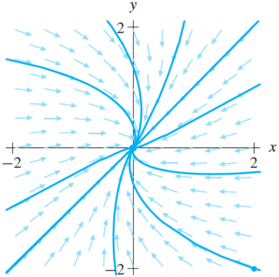

Figure 3.37
After

Systems for Which Every Vector Is an Eigenvector

- Consider
$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mathbf{Y}.$$
- Every vector is an eigenvector.
- Every line that passes through the origin is a solution curve.
- If $a < 0$, then it is a sink whereas if $a > 0$, it is a source.

([PRG], p. 324)

Systems with Zero as an Eigenvalue

- Consider $\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$ where

$$\mathbf{A} = \begin{pmatrix} -3 & 1 \\ 3 & -1 \end{pmatrix}$$

- Eigenvalues: $\lambda_1 = 0, \lambda_2 = -4$.
- Corresponding eigenvectors: $\mathbf{V}_1 = (1, 3), \mathbf{V}_2 = (-1, 1)$.
- The general solution:

$$\begin{aligned} \mathbf{Y} &= k_1 e^{\lambda_1 t} \mathbf{V}_1 + k_2 e^{\lambda_2 t} \mathbf{V}_2 \\ &= k_1 \mathbf{V}_1 + k_2 e^{\lambda_2 t} \mathbf{V}_2. \end{aligned}$$

How can we draw a phase portrait?

([PRG], p. 325)

Systems with Zero as an Eigenvalue

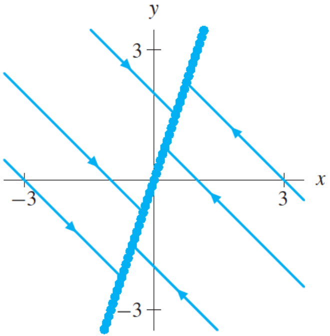

Figure 3.39
Phase portrait for the system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -3 & 1 \\ 3 & -1 \end{pmatrix} \mathbf{Y}.$$

Solutions tend toward the line of equilibrium points.

([PRG], p. 325)

Overview

1. Sect. 3.5 Special Cases: Repeated and Zero Eigenvalues
   - A System with Repeated Eigenvalues
   - The Form of the General Solution
   - Qualitative Analysis of Systems with Repeated Eigenvalues
   - Systems for Which Every Vector Is an Eigenvector
   - Systems with Zero as an Eigenvalue
   - Homework

What's next: Sect. 3.6 Second-Order Linear Equations

Homework

- Suggested Exercises (optional): 1, 3, 5, 7, 13, 15, 17, 23
- Homework Exercises (required to submit): 1, 5, 15, 17

References

Paul Blanchard, Robert L. Devaney, Glen R. Hall
Differential Equations, fourth edition.