Chapter 3 Linear Systems

Sect. 3.6 Second-Order Linear Equations

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Overview

Second-Order Equations versus First-Order Systems

A harmonic oscillator can be modeled by:

1. second-order DE

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0$$

   where $m > 0, k > 0, b \geq 0$, or

2. the corresponding linear system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix} \mathbf{Y}.$$

   where $p = b/m, q = k/m$.

([PRG], p. 330)

Second-Order Equations versus First-Order Systems

1. To solve the second-order DE $m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0$, we guess $y(t) = e^{\lambda t}$. Then it boils down to solving

$$\lambda^2 + \frac{b}{m}\lambda + \frac{k}{m} = 0.$$

2. To solve $d\mathbf{Y}/dt = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix} \mathbf{Y}$, we end up solving

$$\lambda^2 + p\lambda + q = 0.$$

Since in either case we end up solving the same characteristic polynomial, there is no real difference between these two methods.

A Classification of Harmonic Oscillators

Consider

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0.$$

We have four cases depending on damping ▸ Harmonic Oscillator

1. undamped
2. damped
    1. underdamped
    2. overdamped
    3. critically damped.

Can you tell which one is in which case for sure?

$$1)\frac{d^2y}{dt^2} + 0.2\frac{dy}{dt} + 1.01y = 0$$

$$2)\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = 0$$

$$3)\frac{d^2y}{dt^2} + 2\sqrt{2}\frac{dy}{dt} + 2y = 0.$$

The Undamped Harmonic Oscillator

Consider

$$m\frac{d^2y}{dt^2} + ky = 0.$$

The characteristic polynomial:

$$m\lambda^2 + k = 0.$$

Therefore,

$$y(t) = k_1 \cos \omega t + k_2 \sin \omega t$$

while the vector form:

$$\boldsymbol{Y}(t) = k_1 \begin{pmatrix} \cos \omega t \\ -\omega \sin \omega t \end{pmatrix} + k_2 \begin{pmatrix} \sin \omega t \\ \omega \cos \omega t \end{pmatrix}$$

Undamped Harmonic Oscillator

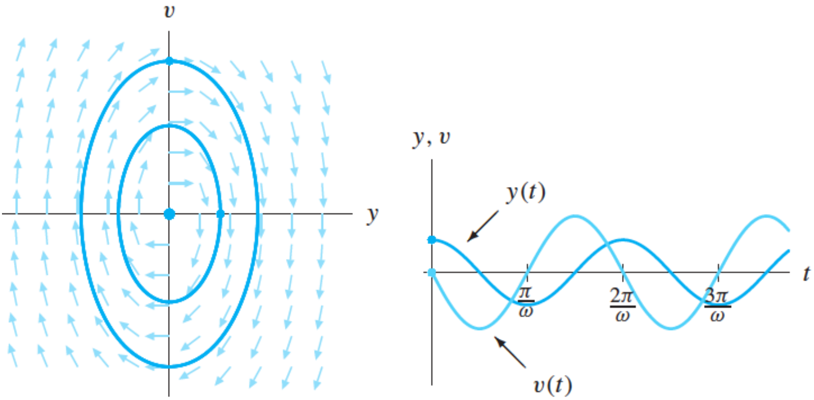

Figure 3.41

Harmonic Oscillator with Damping

- Consider
$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0.$$
- The characteristic polynomial:
$$m\lambda^2 + b\lambda + k = 0.$$
- So
$$\lambda = \frac{b^2 \pm \sqrt{b^2 - 4mk}}{2m}$$
- The type of equilibrium points depends on the sign of $b^2 - 4mk$.

An Underdamped Harmonic Oscillator

If $b^2 - 4mk < 0$, then we get two complex eigenvalues. As the part of the eigenvalues is negative, the origin is a spiral sink.

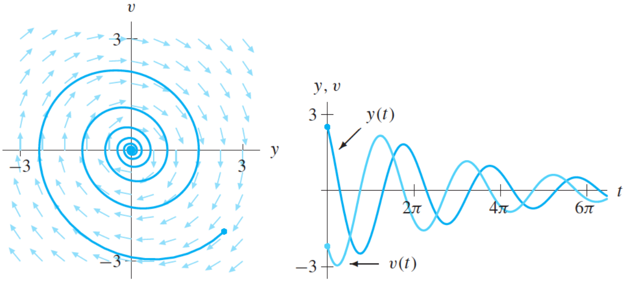

Figure 3.42
Solution in the phase plane and the $y(t)$- and $v(t)$-graphs for the underdamped harmonic oscillator

$$\frac{d^2y}{dt^2} + 0.2\frac{dy}{dt} + 1.01y = 0.$$

An Overdamped Oscillator

If $b^2 - 4mk > 0$, then we get two real, negative eigenvalues. So the origin is a sink.

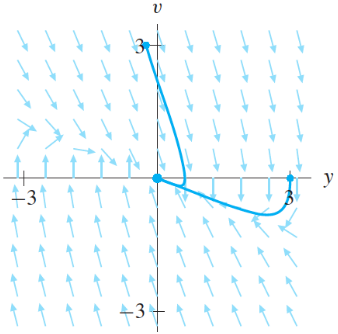

Figure 3.43
The direction field and two solution curves for

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & -3 \end{pmatrix} \mathbf{Y}.$$

One solution curve has initial condition $(y_0, v_0) = (3, 0)$, and the other solution curve has initial condition $(y_0, v_0) = (-0.25, 3)$.

A Critically Damped Oscillator

If $b^2 - 4mk = 0$, then we get a repeated negative eigenvalue.

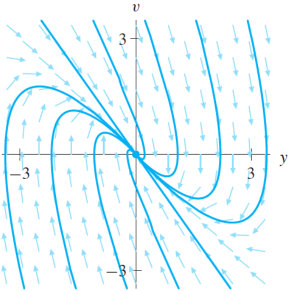

Figure 3.34
Direction field and solution curves for

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -2 & -2\sqrt{2} \end{pmatrix} \mathbf{Y}.$$

Note that the solution curves approach the origin tangent to the line

$$v = -\sqrt{2}\, y$$

of eigenvectors.

Summary

Harmonic Oscillator

Consider

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0.$$

- If $b = 0$, the oscillator is undamped, and the origin is a center.
- If $b > 0$ and $b^2 - 4mk < 0$, the oscillator is underdamped, and the origin is a spiral sink.
- If $b > 0$, $b^2 - 4mk > 0$, the oscillator is overdamped, and the origin is a sink
- If $b > 0$, $b^2 - 4mk = 0$, then the oscillator is critically damped, and the system has exactly one eigenvalue, which is negative.

([PRG], p. 341)

Overview

1. Sect. 3.6 Second-Order Linear Equations
   - Second-Order Equations versus First-Order Systems
   - A Classification of Harmonic Oscillators
   - Summary
   - Homework

What's next: Sect. 3.7 The Trace-Determinant Plane

Homework

- Suggested Exercises (optional): 1-5 odd, 7-11 odd, 13, 31, 39
- Homework Exercises (required to submit): 1, 3, 7, 9, 13, 31

References

Paul Blanchard, Robert L. Devaney, Glen R. Hall
Differential Equations, fourth edition.