# Chapter 3 Linear Systems

## Sect. 3.7 The Trace-Determinant Plane

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# Overview

# Trace and Determinant

Recall

**Linearization Theorem**

Suppose $y_0$ is an equilibrium point of $dy/dt = f(y)$ where $f$ is a continuously differentiable function. Then

- if $f'(y_0) < 0$, then $y_0$ is a sink;
- if $f'(y_0) > 0$, then $y_0$ is a source; or
- if $f(y_0) = 0$, then it is inconclusive.

It was simple and beautiful, but when it comes to systems...
([PRG], p. 348)

# Trace and Determinant

Table 3.1
Partial table of linear systems.

| Type | Eigenvalues | Phase Plane | Type | Eigenvalues | Phase Plane |
|---|---|---|---|---|---|
| Saddle | $\lambda_1 < 0 < \lambda_2$ |  | Spiral Sink | $\lambda = a \pm ib$<br>$a < 0, b \neq 0$ |  |
| Sink | $\lambda_1 < \lambda_2 < 0$ |  | Spiral Source | $\lambda = a \pm ib$<br>$a > 0, b \neq 0$ |  |
| Source | $0 < \lambda_1 < \lambda_2$ |  | Center | $\lambda = \pm ib$<br>$b \neq 0$ |  |

It does not give us a clear understanding of the big picture.

# Trace and Determinant

- We want to find out two essential factors besides eigenvalues, that determine the type of equilibrium points.
- Consider

$$\frac{d\boldsymbol{Y}}{dt} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \boldsymbol{Y}.$$

- Then the characteristic polynomial for $\boldsymbol{A}$ is

$$\lambda^2 - (a+d)\lambda + (ad - bc).$$

- Since the type of equilibrium points only depends on the characteristic polynomial, the trace $T = a + d$ and the determinant $D = ad - bc$ are really essential factors.

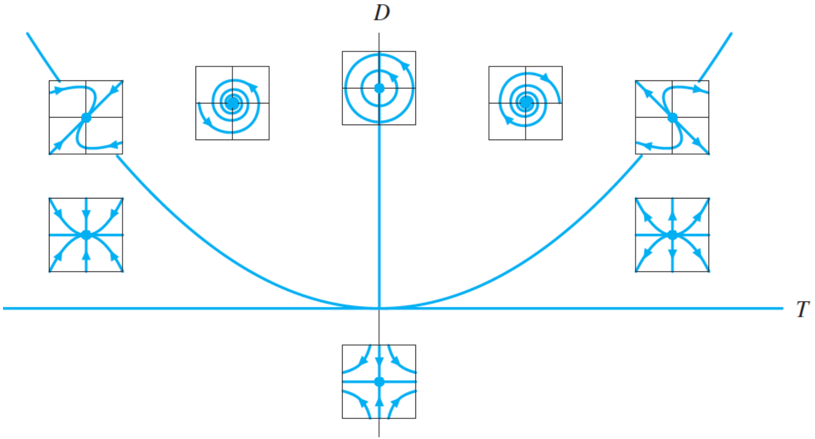
# Trace and Determinant

- Specifically, we can write eigenvalues in terms of $T, D$:

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}.$$

- Now we can paint the big picture based on the trace $T$ and the determinant $D$. (Detail 1)

# The Trace-Determinant Plane



# The Trace-Determinant Palne

Unfortunately, even the big picture is not complete.

- Along the repeated-root parabola we have repeated eigenvalues, but we cannot determine whether we have one or many linearly independent eigenvectors, only based on the trace and determinant.
- We cannot determine the direction in which solutions wind about the origin if $T^2 - 4D < 0$, only based on the trace and determinant.

([PRG], p. 351)

# The Harmonic Oscillator

We can also paint the big picture for the harmonic oscillator. (Detail 2)

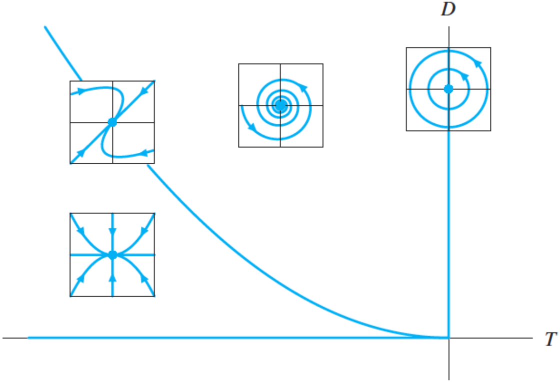



Figure 3.49

# Navigating the Trace-Determinant Plane

- If we change parameters of $d\mathbf{Y}/dt = \mathbf{AY}$, then the trace and the determinant vary, and
- usually the basic behavior of solutions remains more or less the same.
- However, if we pass over the following critical loci, the system undergoes a bifurcation.
  - the positive $D$- aixs,
  - $T$-aixs,
  - the repeated-root parabola.

(Detail 3)
([PRG], p. 353)

# A One-Parameter Family of Linear Systems

As an example, consider an one-parameter family of systems (Detail 4)

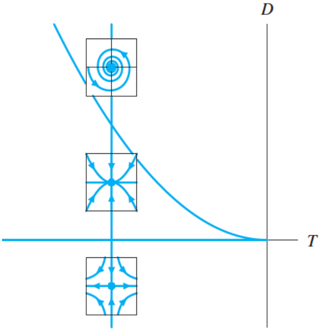



**Figure 3.50**
Motion in the trace-determinant plane corresponding to the one-parameter family of systems

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}, \quad \text{where } \mathbf{A} = \begin{pmatrix} -2 & a \\ -2 & 0 \end{pmatrix}.$$

([PRG], p. 354)

# Overview

1. Sect. 3.7 The Trace-Determinant Plane
   - Trace and Determinant
   - The Trace-Determinant Plane
   - The Harmonic Oscillator
   - Navigating the Trace-Determinant Plane
   - A One-Parameter Family of Linear Systems
   - Homework

What's next: Sect. 3.8 Linear Systems in Three Dimensions

# Homework

- Suggested Exercises (optional): 3-7 odd, 9, 11,
- Homework Exercises (required to submit): 3-5 (a,c), 11 (a,b),

## References

Paul Blanchard, Robert L. Devaney, Glen R. Hall
Differential Equations, fourth edition.