Chapter 3 Linear Systems

Sect. 3.8 Linear Systems in Three Dimensions

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Overview

Linear Systems in Three Dimensions

- So far, we have studied linear systems with two dependent variables.
- It has the *linearity principle*.
- The nature of phase planes are determined by *eigenvalues and eigenvectors*.
- Question) How about three-dimensional systems?

Linear Independence and the Linearity Principle

The Linearity Principle

- Consider

$$\frac{d\boldsymbol{Y}}{dt} = \boldsymbol{A}\boldsymbol{Y}, \quad \boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

- If $\boldsymbol{Y}_1(t)$, $\boldsymbol{Y}_2(t)$ are solutions, then $k_1\boldsymbol{Y}_1(t) + k_2\boldsymbol{Y}_2(t)$ is also a solution for any constant k_1, k_2.
- Moreover, if $\boldsymbol{Y}_1$, $\boldsymbol{Y}_2$, $\boldsymbol{Y}_3$ are linearly independent solutions, then the general solution:

$$k_1\boldsymbol{Y}_1 + k_2\boldsymbol{Y}_2 + k_3\boldsymbol{Y}_3.$$

([PRG], p. 360)

Eigenvalues and Eigenvectors

- Consider

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y} = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \mathbf{Y}.$$

- Eigenvalues: $\lambda_1 = -3, \lambda_2 = -1, \lambda_3 = -2$
- Eigenvectors: (Detail 1) $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ respectively.
- Solutions:

$$\mathbf{Y}_1 = e^{-3t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{Y}_2 = e^{-t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{Y}_3 = e^{-2t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

- The general solution:

$$k_1 \mathbf{Y}_1 + k_2 \mathbf{Y}_2 + k_3 \mathbf{Y}_3$$

([PRG], p. 363)

Eigenvalues and Eigenvectors

So, for the example,

- the coordinate axes form straight-line solutions, and
- all three of the eigenvalues are negative.

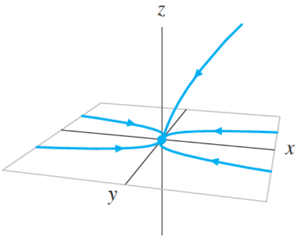

Figure 3.58
Phase space for $d\mathbf{Y}/dt = \mathbf{AY}$ for the diagonal matrix $\mathbf{A}$.

Partially Decoupled System

- Consider
$$\frac{d\boldsymbol{Y}}{dt} = \boldsymbol{BY} = \begin{pmatrix} 0.1 & -1 & 0 \\ 1 & 0.1 & 0 \\ 0 & 0 & -0.2 \end{pmatrix} \boldsymbol{Y}$$
.
- This system decouples into
$$\frac{dx}{dt} = 0.1x - y$$
$$\frac{dy}{dt} = x + 0.1y$$
and
$$\frac{dz}{dt} = -0.2z$$
- In the xy-plane, the eigenvalues: $0.1 \pm i$. Along z-axis, zero is a sink.

Partially Decoupled System

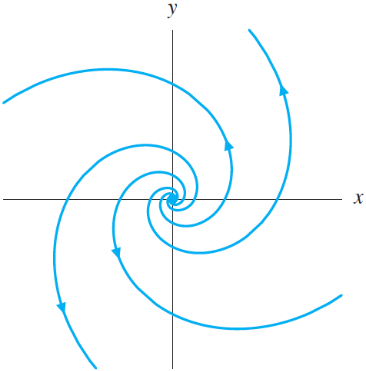

Figure 3.59
Phase plane for xy-system and phase line for z.

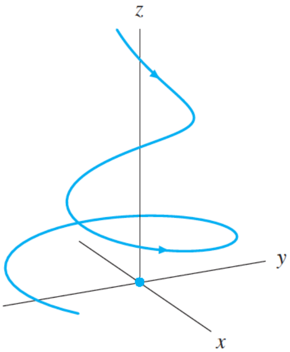

Figure 3.60
Phase space for $d\mathbf{Y}/dt = \mathbf{B}\mathbf{Y}$.

Classification of Three-Dimensional Linear Systems

The most important types of three-dimensional systems: sinks, sources, and saddles.

- We call the equilibrium point a **sink** if all solutions tend toward it as time increases, and a **source** if all solutions tend away from the origin as time increases.
- We call the equilibrium point a **saddle** if, as time increases, some solutions tend toward it while other solutions move away from it.
  (This classification is not complete.)

([PRG], p. 367)

Sinks and Sources

The possibilities for a sink (a source) are to have

- three real, negative (positive) eigenvalues
- one real, negative (positive) eigenvalue and two complex eigenvalues with negative (positive) real parts

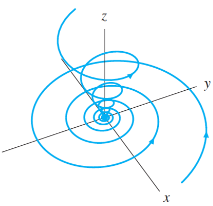

Figure 3.61
Example phase space for spiral sink.

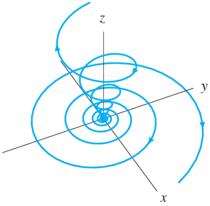

Figure 3.62
Example phase space for spiral source.

Saddles

The possibilities for a saddles is to have

- if all the eigenvalues are real,
  - one positive and two negative eigenvalues or
  - two positive and one negative eigenvalues;
- if we have only one real eigenvalues and the other two are a complex conjugate,
  - the real negative eigenvalue and the positive real parts of the complex eigenvalues, or
  - the real positive eigenvalue and the negative real parts of the complex eigenvalues.

Saddles

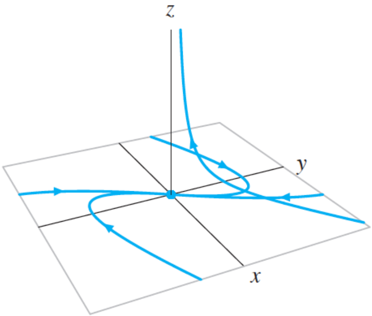

Figure 3.63
Example of a saddle with one positive and two negative eigenvalues.

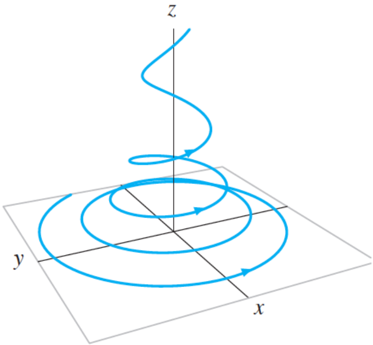

Figure 3.64
Example of a saddle with one real eigenvalue and a complex conjugate pair of eigenvalues.

Overview of Sect. 3.8 Linear Systems in Three Dimensions

What's next: Chapt. 4 Forcing and Resonance

Homework

- Suggested Exercises (optional): 5, 6, 11, 13, 17 except (d)
- Homework Exercises (required to submit): 5, 13, 17 except (d).

Overview of Chapt. 3

1. Properties of Linear Systems and the Linearity Principle
2. Straight-Line Solutions
3. Phase Portraits for Linear Systems with Real Eigenvalues
4. Complex Eigenvalues
5. Special Cases: Repeated and Zero Eigenvalues
6. Second-Order Linear Equations
7. The Trace-Determinant Plane
8. Linear Systems in Three Dimensions

References

Paul Blanchard, Robert L. Devaney, Glen R. Hall
Differential Equations, fourth edition.