Chapter 4 Forcing and Resonance

Sect. 4.1 Forced Harmonic Oscillator

Jeaheang(Jay) Bang

Rutgers University

j.bang@rutgers.edu

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Overview of Chapt 4

- We know how to solve harmonic oscillators

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = 0$$

where $p \geq 0, q > 0$. (Detail 1)

- But what if we have a forcing term? ▸ Forced Harmonic Oscillator

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = f(t)$$

- In this chapter, we will study how to solve this for typical forcing functions.

([PRG], p.387)

Overview of Chapt 4 Forcing and Resonance

- ▸ Tacoma Bridge ▸ Breaking a Glass Using Resonance
- In this Chapter, we will also discuss the phenomenon of resonance and study recent models that explain the collapse of the Tacoma Bridge.

([PRG], p.387)

Overview of Chapt 4

1. Forced Harmonic Oscillators
2. Sinusoidal Forcing
3. Undamped Forcing and Resonance
4. Amplitude and Phase of the Steady State
5. The Tacoma Narrows Bridge

Overview of Sect. 4.1 Forced Harmonic Oscillator

An Equation for the Forced Harmonic Oscillator

- Now consider force acting on a harmonic oscillator.
- The new equation is

$$m\frac{d^2y}{dt^2} = -ky - b\frac{dy}{dt} + f(t).$$

or

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = g(t).$$

- This is a second-order, linear, constant-coefficient, **nonhomogeneous**, nonautonomous equation.
- Question) What is our strategy to solve this kind of equations?

([PRG], p.388)

The Extended Linearity Principle

Consider

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = g(t)$$

and its associated homogeneous equation

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = 0$$

([PRG], p.390)

The Extended Linearity Principle

The Extended Linearity Principle

1. Suppose $y_p(t)$ is a particular solution of the nonhomogeneous equation and $y_h(t)$ is a solution of the associated homogeneous equation. Then $y_h(t) + y_p(t)$ is also a solution of the nonhomogeneous equation.
2. Suppose $y_p(t)$ and $y_q(t)$ are two solutions of the nonhomogeneous equation. Then $y_p(t) - y_q(t)$ is a solution of the associated homogeneous equation.

The Extended Linearity Principle

The Extended Linearity Principle (cont.)

Therefore, if $k_1 y_1(t) + k_2 y_2(t)$ is the general solution of the homogeneous equation, then

$$k_1 y_1(t) + k_2 y_2(t) + y_p(t)$$

is the general solution of the nonhomogeneous equation.

Why? (Detail 2)

The Extended Linearity Principle

Algorithm for finding the general solution of equations for forced harmonic oscillators:

1. Find the general solution of the associated homogeneous second-order equation
2. Find one particular solution of the nonhomogeneous second-order equation
3. Add the results to obtain the general solution of the forced equation.

An Example of the Method of Undetermined Coefficients

Consider

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = e^{-t}.$$

1. Find the general solution to $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 0$. The general solution (Detail 3) : $y_h(t) = k_1 e^{-2t} + k_2 e^{-3t}$
2. Find a particular solution to $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = e^{-t}$ by guessing. (Detail 4) We obtain $y_p(t) = e^{-t}/2$.
3. So the general solution:

$$y(t) = k_1 e^{-2t} + k_2 e^{-3t} + \frac{e^{-t}}{2}$$

([PRG], p.393)

Qualitative Implications of the Extended Linearity Principle

The general solution is

$$y(t) = k_1 e^{-2t} + k_2 e^{-3t} + \frac{e^{-t}}{2}$$

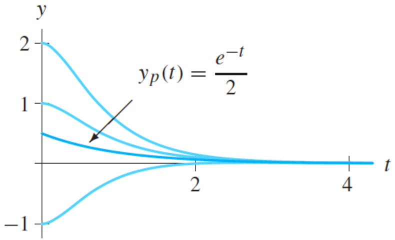

Figure 4.2
Several solutions of

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = e^{-t}.$$

Note that all of the graphs are asymptotic to $y_p(t) = e^{-t}/2$ as t increases.

Let us generalize our observation. ([PRG], p.392)

Qualitative Implications of the Extended Linearity Principle

Consider

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = g(t)$$

where $q > 0, p \geq 0$.

- The general solution is

$$k_1 y_1(t) + k_2 y_2(t) + y_p(t)$$

- We know $k_1 y_1(t) + k_2 y_2(t) \to 0$ as $t \to \infty$.
- So for large t,

$$k_1 y_1(t) + k_2 y_2(t) + y_p(t) \approx y_p(t)$$

([PRG], p.392)

Qualitative Implications of the Extended Linearity Principle

In other words, the initial conditions have no effect on the long-term behavior of the solutions

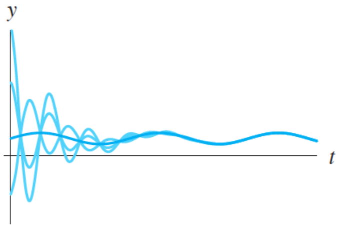

Figure 4.1
Typical graphs of solutions of a forced harmonic oscillator equation with nonzero damping ($p > 0$). The solution $y_p(t)$ is shown in dark blue.

Qualitative Implications of the Extended Linearity Principle

In the above discussion,

Steady-State Solution

- the particular solution $y_p(t)$ is called the **forced response** or **steady-state solution**.
- the homogeneous solution y_h is called the **natural response**

Te discussion above can be restated as:

All Solutions of a forced, damped harmonic oscillator approach the steady-state solution because the natural response dies out, leaving only the forced response

Second Guessing

Question) Does our guessing technique always work?

- Consider

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = e^{-2t}.$$

- We are stuck in our first guess for finding a particular solution. (Detail 5)
- Guess $y_p(t) = kte^{-2t}$. (Detail 6)

([PRG], p.395)

Exercises

Let us solve more examples that we can apply the guessing technique to.

- Solve
$$\frac{d^2y}{dt^2} + 4y = -3t^2 + 2t + 3.$$
(Homework # 31)
- Solve
$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = e^{-t} + 4.$$
(Homework # 37)

Overview of Sect. 4.1 Forcing and Resonance

1. Sect. 4.1 Forced Harmonic Oscillators
   - An Equation for the Forced Harmonic Oscillator
   - The Extended Linearity Principle
   - An Example of the Method of Undetermined Coefficients
   - Qualitative Implications of the Extended Linearity Principle
   - Second Guessing
   - Homework

What's next: Sect. 4.2 Sinusoidal Forcing

Homework

- Suggested Exercises (optional): 1-5 odd, 9, 11, 13, 15, 19, 21, 31, 33, 37, 39
- Homework Exercises (required to submit): 1, 3, 9, 13, 19, 21, 31, 33, 37

References

Paul Blanchard, Robert L. Devaney, Glen R. Hall
Differential Equations, fourth edition.