Chapter 4 Forcing and Resonance

Sect. 4.2 Sinusoidal Forcing

Jeaheang(Jay) Bang

Rutgers University

j.bang@rutgers.edu

Tue. Aug 1, 2017

Overview

A Sound Striking a Glass | Forced Oscillation Resonance

Sinusoidal Forcing of a Damped Harmonic Oscillator

- Consider

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = \sin t.$$

- The general solution to the unforced equation is

$$y_h(t) = k_1 e^{-t} \cos t + k_2 e^{-t} \sin t.$$

- In order to find a particular solution...

([PRG], p. 403)

Complexification

- consider

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = e^{it}.$$

- Why? (Detail 1)
- We guess $y_c(t) = ae^{it}$. (Detail 2) Substituting this into the equation, we get

$$y_c(t) = \frac{1-2i}{5}(\cos t + i \sin t).$$

- So taking the imaginary part, we obtain a particular solution of the original equation

$$y_p(t) = -\frac{2}{5}\cos t + \frac{1}{5}\sin t.$$

([PRG], p. 404)

Complexification

Hence the general solution of

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = \sin t$$

is

$$y(t) = k_1 e^{-t} \sin t + k_2 e^{-t} \cos t + \left(-\frac{2}{5}\cos t + \frac{1}{5}\sin t\right).$$

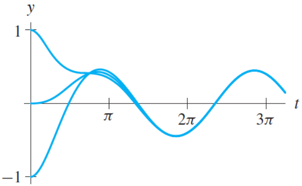

Figure 4.9
Several solutions of the equation

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = \sin t.$$

Question) Can we find some relation between the forcing term and the particular solution?

Complexification

- Recall $y_c = ae^{it}$ and $a = \frac{1-2i}{5}$.
- We write

$$a = |a|e^{i\theta} \quad \text{for some } \theta.$$

- So

$$y_c = |a|e^{i\theta}e^{it} = |a|e^{i(t+\theta)} = |a|(\cos(t+\theta) + i\sin(t+\theta))$$

- Thus

$$y_p(t) = |a|\sin(t+\theta) = |a|\cos(t-(\theta+90^\circ))$$

- We compare this to the forcing term

$$\sin t = \cos(t - 90^\circ).$$

Complexification

- Therefore, the forcing term $\cos(t - 90^\circ)$ has some relation with the particular solution

$$|a|\cos(t-\phi), \quad \phi = \theta + 90^\circ$$

- But it is still not clear how $|a|, \theta$ are determined.
- $|a|$ is called the **amplitude** and ϕ is called the **phase angle.**
- We will discuss this in depth in Sect. 4.4 Amplitude and Phase of the Steady State.

The Phase Portrait of a Forced Harmonic Oscillator

Now let us draw the phase portrait of a forced harmonic oscillator

- Consider
$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 10y = 4\cos 2t.$$
- Using the Extended Linearity Principle, the general solution is
$$y(t) = k_1 e^{-t}\cos 3t + k_2 e^{-t}\sin 3t + \frac{2\sqrt{13}}{13}\cos(2t+\phi)$$
for some phase angle ϕ.

([PRG], p. 407)

The Phase Portrait of a Forced Harmonic Oscillator

Because we find $y(t)$, we can compute $v = \frac{dy}{dt}$ and graph solutions in both ty-plane and yv-plane.

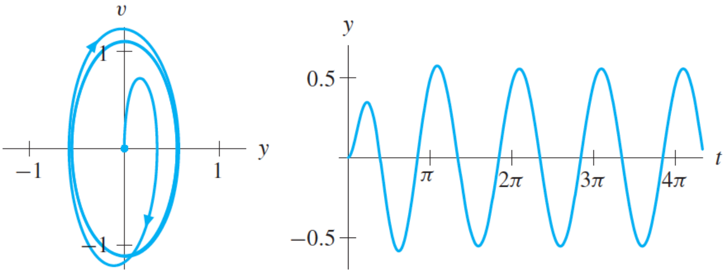

Figure 4.10
Graph of the solution of $d^2y/dt^2 + 2dy/dt + 10y = 4\cos 2t$ with initial conditions $y(0) = y'(0) = 0$ in the yv- and ty-planes.

The Phase Portrait of a Forced Harmonic Oscillator

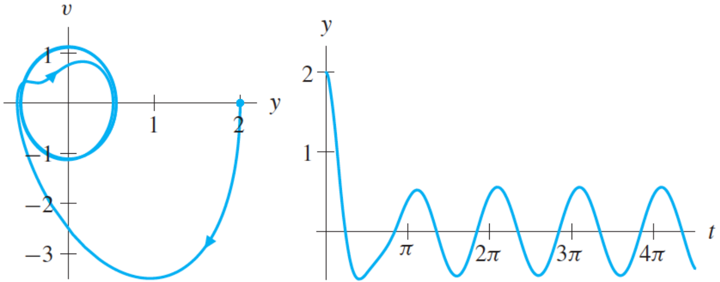

Figure 4.11
Graph of the solution of $d^2y/dt^2 + 2dy/dt + 10y = 4\cos 2t$ with initial conditions $y(0) = 2$ and $y'(0) = 0$ in the yv- and ty-planes.

The Phase Portrait of a Forced Harmonic Oscillator

What if we choose an initial condition far away from the origin? (Detail 3)

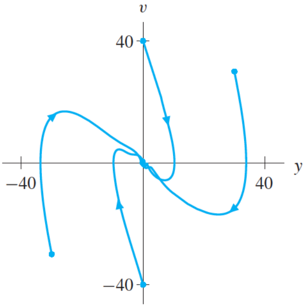

Figure 4.12
Four solutions of
$d^2y/dt^2 + 2dy/dt + 10y = 4\cos 2t$ with initial conditions far from the origin.

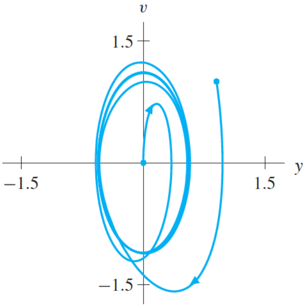

Figure 4.13
Two solutions of
$d^2y/dt^2 + 2dy/dt + 10y = 4\cos 2t$ with initial conditions close to the origin.

Exercise

Homework Exercise # 19)
Find the general solution of

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 20y = e^{-t}\cos t.$$

Overview

What's next: Sect. 4.3 Undamped Forcing and Resonance

Homework

- Suggested Exercises (optional): 1-5 odd, 11, 13, 17, 19,
- Homework Exercises (required to submit): 1-5 odd, 11, 13, and 22 on page 313.

References

Paul Blanchard, Robert L. Devaney, Glen R. Hall
Differential Equations, fourth edition.