Chapter 4 Forcing and Resonance

Sect. 4.3 Undamped Forcing and Resonance

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Overview

An Undamped Harmonic Oscillator with Sinusoidal Forcing

Consider

$$\frac{d^2y}{dt^2} + 2y = \cos \omega t.$$

- The general solution to the unforced equation is

$$y_h(t) = k_1 \cos \sqrt{2}t + k_2 \sin \sqrt{2}t.$$

- Because there is no damping, the solutions oscillate for all time with the **natural frequency** $\sqrt{2}/(2\pi)$.

([PRG], p. 415)

An Undamped Harmonic Oscillator with Sinusoidal Forcing

Consider

$$\frac{d^2y}{dt^2} + 2y = \cos \omega t.$$

- Complexifying the equation, we obtain a particular solution (Detail 1)

$$y_p(t) = \frac{1}{2 - \omega^2} \cos \omega t.$$

  The solution oscillates with the frequency $\omega/(2\pi)$
- Therefore, by the extended linearity principle,

$$y(t) = k_1 \cos \sqrt{2}t + k_2 \sin \sqrt{2}t + \frac{1}{2 - \omega^2} \cos \omega t.$$

Question) How does the graph look like?
([PRG], p. 415)

Qualitative Analysis

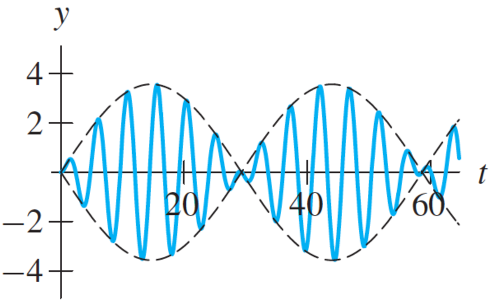

Figure 4.15
Solution of $d^2y/dt^2 + 2y = \cos \omega t$
for $\omega = 1.2$, $y(0) = y'(0) = 0$.

Qualitative Analysis

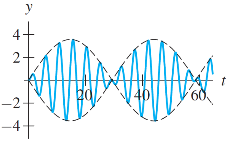

Figure 4.15
Solution of $d^2y/dt^2 + 2y = \cos \omega t$ for $\omega = 1.2$, $y(0) = y'(0) = 0$.

- The amplitudes of the oscillations increase and decrease in a very regular pattern. This phenomenon is called **beating**.
- Question) Why is this phenomenon happening? Can we determine the frequency of the beats?

Qualitative Analysis

Actually assigning $y(0) = y'(0) = 0$, we can obtain (Detail 2)

$$y(t) = a(\cos\omega t - \cos\sqrt{2}t)$$
$$= -2a\left[\sin\left(\frac{\omega+\sqrt{2}}{2}\right)t\right]\left[\sin\left(\frac{\omega-\sqrt{2}}{2}\right)t\right]$$

So the graph makes sense. (Detail 3)

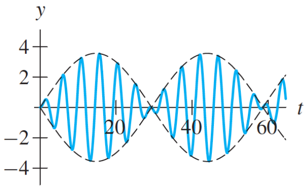

Figure 4.15
Solution of $d^2y/dt^2 + 2y = \cos\omega t$ for $\omega = 1.2$, $y(0) = y'(0) = 0$.

Dependence of Amplitude on Forcing Frequency

Recall the general solution is

$$y(t) = k_1 \cos \sqrt{2}t + k_2 \sin \sqrt{2}t + \frac{1}{2-\omega^2} \cos \omega t.$$

Every solution contains $\frac{1}{2-\omega^2} \cos \omega t$ and the amplitude of this is given by $1/|2-\omega^2|$.

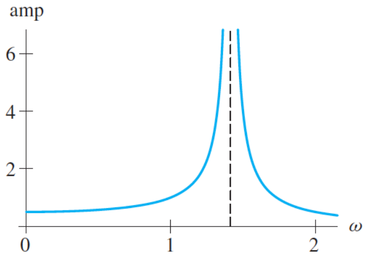

Figure 4.17
Graph of $1/|2-\omega^2|$, the amplitude of the oscillations of solutions of

$$\frac{d^2y}{dt^2} + 2y = \cos \omega t$$

for large t.

Dependence of Amplitude on Forcing Frequency

So, if ω is very large,

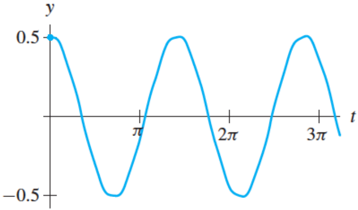

Figure 4.18
Solution of

$$\frac{d^2y}{dt^2} + 2y = \cos 10t$$

for initial conditions $y(0) = 0.5$, $y'(0) = 0$.

Dependence of Amplitude on Forcing Frequency

So, if ω is close to $\sqrt{2}$,

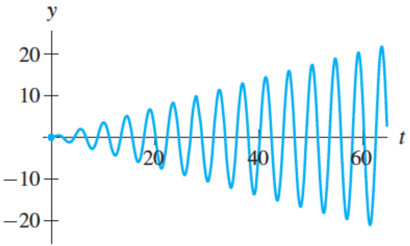

Figure 4.19
Solution of

$$\frac{d^2y}{dt^2} + 2y = \cos 1.4t$$

with $y(0) = y'(0) = 0$.

Resonance

What if $\omega = \sqrt{2}$?

- In this case, the forcing is called **resonant forcing**, and
- the oscillator is said to be in **resonance**.

Then the general solution is (Detail 4)

$$k_1 \cos \sqrt{2}t + k_2 \sin \sqrt{2}t + \frac{1}{2\sqrt{2}} t \sin \sqrt{2}t.$$

So for large t, the term

$$\frac{1}{2\sqrt{2}} t \sin \sqrt{2}t$$

dominates the other terms.
([PRG], p. 421)

Resonance

Therefore, the graphs make sense.

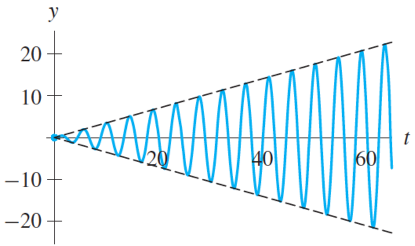

Figure 4.21
Solution of

$$\frac{d^2y}{dt^2} + 2y = \cos\sqrt{2}\,t$$

with $y(0) = y'(0) = 0$. The dashed lines are $y = \pm t/(2\sqrt{2})$.

Exercises

Find the general solution to

3

$$\frac{d^2y}{dt^2} + 4y = -\cos\frac{t}{2}$$

19

$$\frac{d^2y}{dt^2} + 12y = 3\cos 4t + 2\sin t$$

Overview

1. Sect. 4.3 Undamped Forcing and Resonance
   - An Undamped Harmonic Oscillator with Sinusoidal Forcing
   - Qualitative Analysis
   - Resonance
   - Homework

Question) How can we avoid resonance?
What's next: Sect. 4.4 Amplitude and Phase of the Steady State

Homework

- Suggested Exercises (optional): 1-7 odd, 9,11, 15-17 odd, 19, 21
- Homework Exercises (required to submit): 1, 3, 9, 11, 15, 19 except (d)

References

Paul Blanchard, Robert L. Devaney, Glen R. Hall
Differential Equations, fourth edition.