Chapter 4 Forcing and Resonance

Sect. 4.4 Amplitude and Phase of the Steady State

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Overview

We studied $\frac{d^2y}{dt^2} + 2y = \cos\omega t$. The conclusion was that if $\omega = \sqrt{2}$ then resonance happens. What should we do to avoid resonance? (Detail 1)

Amplitude and Phase

Consider

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = \cos\omega t$$

Using the technique of complexification, we get (Detail 2)

$$y_p = A\cos(\omega t + \phi), \quad A = \frac{1}{\sqrt{(q-\omega^2)^2 + p^2\omega^2}}, \quad \tan\phi = \frac{-p\omega}{q-\omega^2}.$$

([PRG], p. 428)

Amplitude and Phase

In order to understand the behavior of the amplitude, we fix q and choose several values of p. Then A is a function of ω.

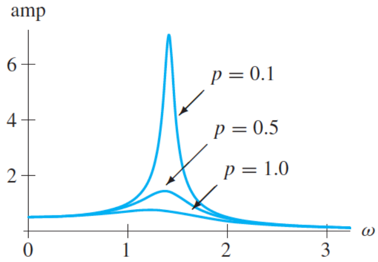

Figure 4.22
Graphs of the amplitude

$$A = \frac{1}{\sqrt{(q-\omega^2)^2 + p^2\omega^2}}$$

with $q = 2$ and $p = 0.1$, $p = 0.5$, and $p = 1.0$.

([PRG], p. 428)

Amplitude and Phase

Or we fix q or ω regarding A as a function of p, ω or of p, q.

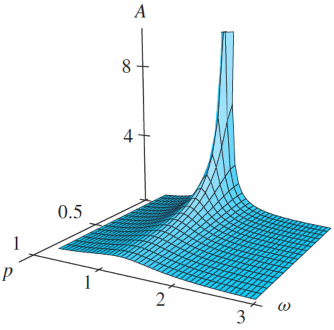

Figure 4.23
Graph of amplitude A as a function of p and ω for $q = 2$.

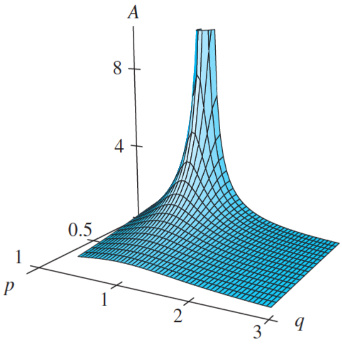

Figure 4.24
Graph of amplitude A as a function of p and q for $\omega = 1$.

([PRG], p. 428)

Amplitude and Phase

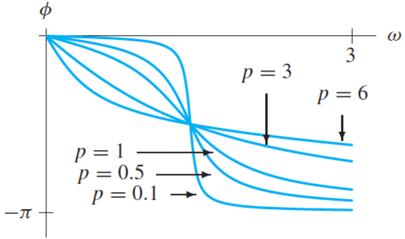

Figure 4.25
Phase angle ϕ for $q = 2$ with $p = 0.1$, $p = 0.5$, $p = 1.0$, $p = 3.0$, and $p = 6.0$. Note that the phase angle ϕ stays close to zero for small ω if p is small. The graph of ϕ has a large slope when it passes through $-\pi/2 = -90°$ for $\omega = \sqrt{q}$. For ω large, ϕ is asymptotic to $-\pi$.

([PRG], p. 428)

An Application for Washing Machines

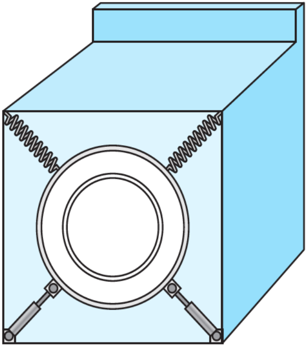

Figure 4.26
Schematic of a front loading washing machine with drum springs and dampers holding the drum.

Dashpots of Washing Machines, 1:30

([PRG], p.432)

An Application for Washing Machines

- Goal: Keep the washing machine from shaking too violently at the resonant frequency.
- Large damping is not a solution because stiff dashpots connect the drum rigidly to the casing.
- A better solution is to install a dashpot that provides a damping coefficient that can be adjusted according to the frequency of the forcing.
- We want to have large damping if the drum has low frequency and low damping if the drum has high frequency.

([PRG], p.432)

An Application for Washing Machines

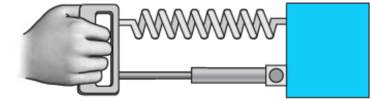

Figure 4.27
Mass attached to a handle by a spring and dashpot. The mass and handle are free to move horizontally.

Let $y(t)$ denote the horizontal position of the mass and $z(t)$ the horizontal position of the handle at time t. The equation governing the motion is

$$\frac{d^2y}{dt^2} = -k(y - z - y_0) - b\left(\frac{dy}{dt} - \frac{dz}{dt}\right).$$

([PRG], p.432)

An Application for Washing Machines

Assume $z(t) = A\cos\omega t$. Then the equation becomes

$$\frac{d^2y}{dt^2} + b\frac{dy}{dt} + y = A\cos\omega t - b\omega A\sin\omega t.$$

Then a particular solution is (Detail 3)

$$y(t) = A\frac{1-\omega^2+b^2\omega^2}{(1-\omega^2)^2+b^2\omega^2}\cos\omega t + A\frac{b\omega^3}{(1-\omega^2)^2+b^2\omega^2}\sin\omega t.$$

([PRG], p.432)

An Application for Washing Machines

In order to keep the washing machine from shaking violently, we assume

$$b(\omega) = \begin{cases} 10, & \text{if } \omega < 2 \\ 0.1 & \text{if } \omega \geq 2. \end{cases}$$

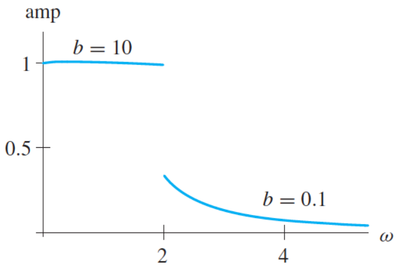

Figure 4.29
Amplitude of the forced response with $b(\omega)$ large for small ω and $b(\omega)$ small for large ω.

Overview

1. Sect. 4.4 Amplitude and Phase of the Steady State
   - Amplitude and Phase
   - An Application for Washing Machines
   - Homework

What's next: Sect. 4.5 The Tacoma Narrows Bridge

Homework

- Homework Exercises (required to submit): 3, 9

References

Paul Blanchard, Robert L. Devaney, Glen R. Hall
Differential Equations, fourth edition.