Chapter 5 Nonlinear Systems

Sect. 5.1 Equilibrium Point Analysis

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Tue. Aug 8, 2017

Overview of Chapt 5 Nonlinear Systems

Can we solve the following systems?

1) $$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -x + (1 - x^2)y$$

2) $$\frac{dx}{dt} = 2x\left(1 - \frac{x}{2}\right) - xy$$
$$\frac{dy}{dt} = 3y\left(1 - \frac{y}{3}\right) - 2xy.$$

What can we do?

Overview of Chapt 5 Nonlinear Systems

- We study nonlinear autonomous systems.
- A nonlinear system can be approximated near an equilibrium point by a linear system. This process is known as **linearization**.
- By looking at where one component of the direction field is zero, we obtain curves called **nullclines**. behaviors of solutions.
- We study special types of nonlinear system: Hamiltonian Systems, Dissipative Systems.

([PRG], p.457)

Overview of Chapt 5 Nonlinear Systems

1. Equilibrium Point Analysis
2. Qualitative Analysis
3. Hamiltonian Systems
4. Dissipative Systems
5. ~~Nonlinear Systems in Three Dimensions~~
6. ~~Periodic Forcing of Nonlinear systems and Chaos~~

Overview of Sect. 5.1 Equilibrium Point Analysis

1. Sect. 5.1 Equilibrium Point Analysis
   - The Van der Pol Equation
   - A Competing Species Model
   - A Nonpolynomial Example
   - Linearization
   - Classification of Equilibrium Points
   - Separatrices
   - When Linearization Fails
   - Homework

The Van der Pol Equation

Study

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -x + (1 - x^2)y.$$

([PRG], p. 458)

The Van der Pol Equation

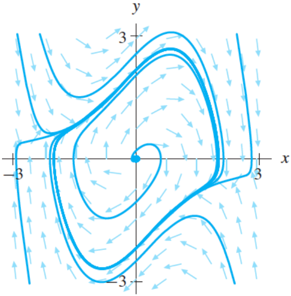

Figure 5.1
Direction field and phase portrait for the Van der Pol system.

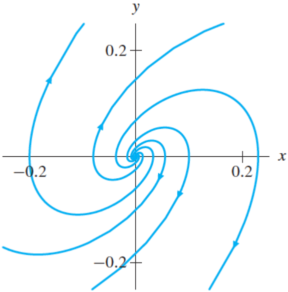

Figure 5.2
Phase portrait for the Van der Pol system near the origin.

We see a picture on the right, that is reminiscent of a spiral source.

The Van der Pol Equation

Let us understand the observation in a qualitative way.

- The equation is

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -x + y - x^2 y.$$

- Assume both x, y are close to 0. Then $x^2 y$ is significantly smaller than either x or y.
- We drop the nonlinear term and are left with

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -x + y. \quad \text{(Detail 1)}$$

- The solutions of the linear system spiral away from the origin
- This technique is called **linearization**.

Exercise

4 Consider the system

$$\frac{dx}{dt} = -x$$
$$\frac{dy}{dt} = -4x^3 + y.$$

Sketch the phase portrait for the linearized system near (0,0).

A Competing Species Model

Consider

$$\begin{aligned}\frac{dx}{dt} &= 2x\left(1-\frac{x}{2}\right) - xy\\ \frac{dy}{dt} &= 3y\left(1-\frac{y}{3}\right) - 2xy.\end{aligned}$$

- Equilibrium points (Detail 2) are (0,0), (0,3), (2,0) and (1,1)
- Solutions with initial conditions in the first quadrant must remain in the first quadrant for all time. Why? (Detail 3)

([PRG], p.459)

A Competing Species Model

We can draw the phase portrait numerically (by Euler's method).

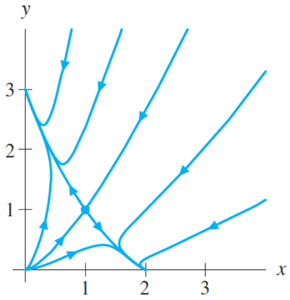

Figure 5.3
Phase portrait for the system

$$\frac{dx}{dt} = 2x\left(1 - \frac{x}{2}\right) - xy$$
$$\frac{dy}{dt} = 3y\left(1 - \frac{y}{3}\right) - 2xy.$$

Do we have to expect to see some solutions leading to (1,1) in nature?

The Van der Pol Equation
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A Competing Species Model

To answer the question, we study the system near the equilibrium point $(1, 1)$ using linearization.

- Near (1,1), both $x - 1, y - 1$ are close to 0.
- We change variables by $u = x - 1, v = y - 1$.
- Then (Detail 4)

$$\frac{du}{dt} = -u - v - u^2 - uv$$
$$\frac{dv}{dt} = -2u - v - 2uv - v^2.$$

A Competing Species Model

Therefore, we approximate the nonlinear system near $(u, v) = (0, 0)$ with the linear system

$$\frac{du}{dt} = -u - v$$
$$\frac{dv}{dt} = -2u - v$$

The eigenvalues are $-1 \pm \sqrt{2}$.

A Competing Species Model

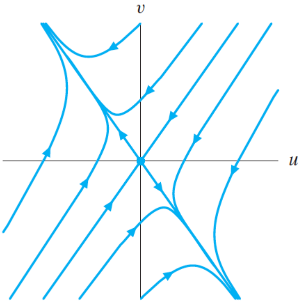

Figure 5.4
Phase portrait for

$$\frac{du}{dt} = -u - v$$
$$\frac{dv}{dt} = -2u - v,$$

the linear approximation of the competitive system near $(x, y) = (1, 1)$, which is the same point as $(u, v) = (0, 0)$.

We expect the phase portrait for the nonlinear system near $(1, 1)$ to look like the uv-phase portrait of the linear system.

A Competing Species Model

Going back to the original system,

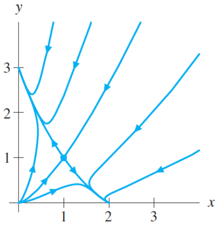

Figure 5.3
Phase portrait for the system

$$\frac{dx}{dt} = 2x\left(1 - \frac{x}{2}\right) - xy$$
$$\frac{dy}{dt} = 3y\left(1 - \frac{y}{3}\right) - 2xy.$$

- we conclude that there are only two curves of solutions in the xy-plane that tend toward $(1, 1)$ as t increases.
- Consequently, we do not expect to see solutions leading to $(1, 1)$ in nature.

A Nonpolynomial Example

- For the previous examples, it is easy to identify which terms are nonlinear.
- But what if we have nonpolynomial term?

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -y - \sin x.$$

- Near the origin, which term should we drop?
- Since $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots,$

([PRG], p.463)

A Nonpolymonial Example

We drop the nonlinear terms and are left with

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -y - x.$$

The eigenvalues are $(-1 \pm \sqrt{3}i)/2$.

A Nonpolynomial Example

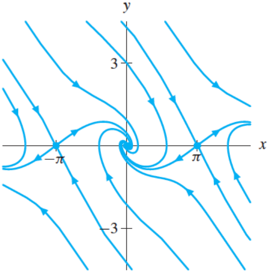

Figure 5.5
Phase portrait for the system

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -y - \sin x.$$

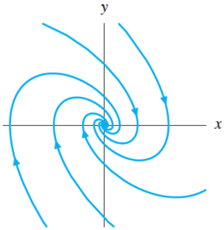

Figure 5.6
Phase portrait for the system

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -y - x.$$

Linearization

Consider

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y).$$

- Suppose (x_0, y_0) is an equilibrium point.
- Introduce new variables $u = x - x_0, v = y - y_0$. Then...

([PRG], p.464)

Linearization

- The new variables u, v satisfy

$$\frac{du}{dt} = f(x_0 + u, y_0 + v)$$
$$\frac{dv}{dt} = g(x_0 + u, y_0 + v).$$

- From calculus,

$$f(x_0 + u, y_0 + v) \approx f(x_0, y_0) + \left[\frac{\partial f}{\partial x}(x_0, y_0)\right] u + \left[\frac{\partial f}{\partial y}(x_0, y_0)\right] v.$$

- Note $f(x_0, y_0) = 0, g(x_0, y_0) = 0.$

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Linearization

Hence the **linearized system** is

$$\frac{dx}{dt} = \left[\frac{\partial f}{\partial x}(x_0, y_0)\right] u + \left[\frac{\partial f}{\partial y}(x_0, y_0)\right] v$$
$$\frac{dv}{dt} = \left[\frac{\partial g}{\partial x}(x_0, y_0)\right] u + \left[\frac{\partial g}{\partial y}(x_0, y_0)\right] v$$

The matrix

$$\begin{pmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\ \frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0) \end{pmatrix}$$

is called **Jacobian matrix** of the system at (x_0, y_0).

More Examples of Linearization

Consider

$$\frac{dx}{dt} = f(x, y) = -2x + 2x^2$$
$$\frac{dy}{dt} = g(x, y) = -3x + y + 3x^2.$$

- Study solutions near $(0, 0)$ or $(1, 0)$.
- In order to find a linearized system, previously, we did (Detail 5)
- But now we compute Jacobian matrix at each point.

([PRG], p.466)

More Examples of Linearization

Compute

$$\begin{pmatrix} \frac{\partial f}{\partial x}(x,y) & \frac{\partial f}{\partial y}(x,y) \\ \frac{\partial g}{\partial x}(x,y) & \frac{\partial g}{\partial y}(x,y) \end{pmatrix} = \begin{pmatrix} -2+4x & 0 -3+6x & 1 \end{pmatrix} =: J(x,y).$$

Evaluate $J(x,y)$ at each equilibrium point;

$$J(0,0) = \begin{pmatrix} -2 & 0 \\ -3 & 1 \end{pmatrix}, \quad J(1,0) = \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix}$$

More Examples of Linearization

- Near $(0,0)$, the phase portrait for the nonlinear system resemble that of the linearized system

$$\frac{d\mathbf{Y}}{dt} = J(0,0)\mathbf{Y} = \begin{pmatrix} -2 & 0 \\ -3 & 1 \end{pmatrix} \mathbf{Y},$$

- Near $(0,1)$, the phase portrait resemble that of the linearized system

$$\frac{d\mathbf{Y}}{dt} = J(0,1)\mathbf{Y} = \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix} \mathbf{Y}.$$

More Examples of Linearization

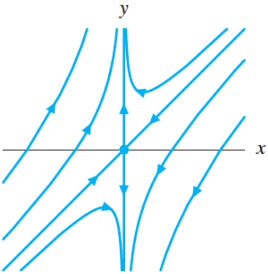


Figure 5.7
Phase portrait for the system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -2 & 0 \\ -3 & 1 \end{pmatrix} \mathbf{Y}.$$

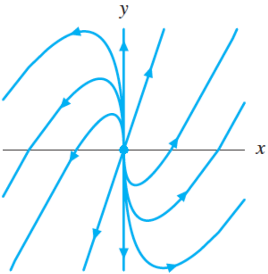
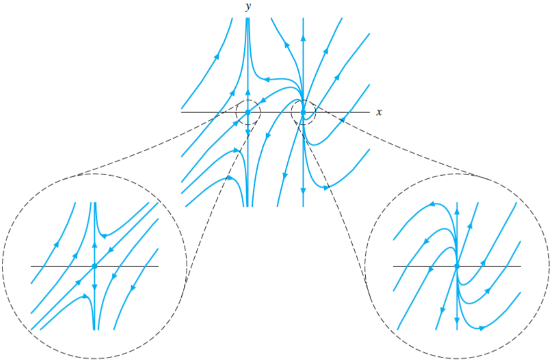


Figure 5.8
Phase portrait for the system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix} \mathbf{Y}.$$

More Examples of Linearization

The solution curves do indeed look like those of the corresponding linearized systems.



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Classification of Equilibrium Points

Consider

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y).$$

Suppose (x_0, y_0) is an equilibrium point and let

$$\boldsymbol{J} = \begin{pmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\ \frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0) \end{pmatrix}.$$

The linearized system is

$$\begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} = \boldsymbol{J} \begin{pmatrix} u \\ v \end{pmatrix}.$$

([PRG], p.468)

Classification of Equilibrium Points

Linearization

- The equilibrium point (x_0, y_0) is a **sink** (**spiral sink**) for the nonlinear system if the origin is a sink (spiral sink) for the linearized system.
- The equilibrium point (x_0, y_0) is a **source** (**spiral source**) for the nonlinear system if the origin is a sink (spiral sink) for the linearized system.
- The equilibrium point (x_0, y_0) is a **saddle** for the nonlinear system if the origin is a saddle for the linearized system.

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Separatrices

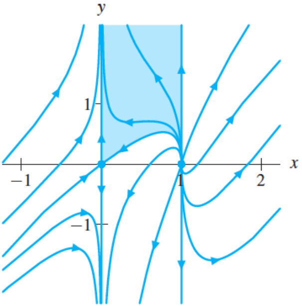


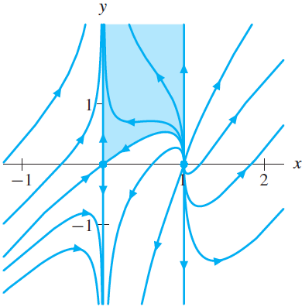
Figure 5.10
Separatrices of (0, 0) for the system

$$\frac{dx}{dt} = -2x + 2x^2$$
$$\frac{dy}{dt} = -3x + y + 3x^2,$$

and regions of the strip between $x = 0$ and $x = 1$ with different long-term behaviors.

([PRG], p.469)

Separatrices



([PRG], p.469)

- The four special solution curve tending toward or moving away from the saddle point are called **separatrices**.
- The two separatrices tending toward the saddle are called **stable separatrices**.
- The two separatrices moving away from the saddle are called **unstable separatrices**.

Exercise

Consider

$$\frac{dx}{dt} = y - (x^2 + y^2)x$$
$$\frac{dy}{dt} = -x - (x^2 + y^2)y.$$

- The origin is an equilibrium point
- Find out its linearized system and classify the equilibrium point.
- Answer: the origin is a center for the linearized system, but...

([PRG], p.470)

When Linearization Fails

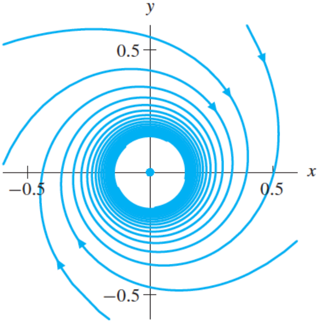


Figure 5.11
The solution curves for the system

$$\frac{dx}{dt} = y - (x^2 + y^2)x$$
$$\frac{dy}{dt} = -x - (x^2 + y^2)y$$

spiral slowly toward the equilibrium point at the origin even though the linearization at the origin is a center.

Question) Why does linearization fail?
([PRG], p.470)

When Linearization Fails

Linearization fails because

> *even very small perturbation cased by the inclusion of the nonlinear terms can turn the center into a spiral sink or a spiral source.*

(Detail 6)

When Linearization Fails

Failure of Linearization

There are only two situations in which the long-term behavior of solutions near an equilibrium point of the nonlinear system and its linearized system can differ.

1. One is when the linearized system is a center
2. The other is when the linearized system has zero as an eigenvalue.

In every other case, the long–term behavior of solutions of the nonlinear system near an equilibrium point is the same as the solutions of its linearization.

Exercise (Bifurcation)

18 Consider

$$\frac{dx}{dt} = x^2 - a$$
$$\frac{dy}{dt} = -y(x^2 + 1)$$

Locate the bifurcation values of a.

Exercise (Bifurcation)

Consider

$$\frac{dx}{dt} = x(-x - y + 70)$$
$$\frac{dy}{dt} = y(-2x - y + a)$$

Find the two bifurcation values of a.

Exercise (Bifurcation)

Consider

$$\frac{dx}{dt} = y - x^2 + \beta$$
$$\frac{dy}{dt} = y - x$$

Locate the bifurcation values of β.

Bifurcation in Nonlinear System

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What's next: Sect. 5.2 Qualitative Analysis

Homework

- Suggested Exercises (optional): 1, 3, 4, 5, 7 except (c), 17, 18, 19, 26,
- Homework Exercises (required to submit): 1, 3, 5, 17

References

Paul Blanchard, Robert L. Devaney, Glen R. Hall
Differential Equations, fourth edition.