Chapter 5 Nonlinear Systems

Sect. 5.2 Qualitative Analysis

Jeaheang(Jay) Bang

Rutgers University

j.bang@rutgers.edu

Wed. Aug 9, 2017

Overview

Competing Species

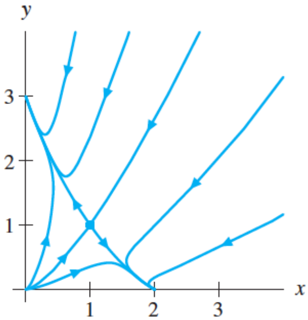

Figure 5.12
Phase portrait for the competing species system

$$\frac{dx}{dt} = 2x\left(1 - \frac{x}{2}\right) - xy$$
$$\frac{dy}{dt} = 3y\left(1 - \frac{y}{3}\right) - 2xy.$$

This computer-generated phase portrait suggests that solutions that do not tend to (1, 1) tend either to (0, 3) or to (2, 0).

To better understand the behavior of solutions, we look more closely at the direction field by using "nullclines". ▸ Nullcline
([PRG], p.478)

Nullclines

Definition

For the system

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y)$$

the x-**nullcline** is the set of points (x, y) where $f(x, y)$ is zero. The y-**nullcline** is the set of points where $g(x, y)$ is zero.

([PRG], p.478)

Nullclines

Consider the competing species example. We find x-nullclines. (Detail 1)

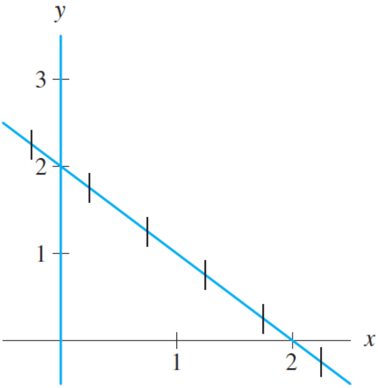

Figure 5.13
The x-nullclines for the system

$$\frac{dx}{dt} = 2x\left(1 - \frac{x}{2}\right) - xy$$
$$\frac{dy}{dt} = 3y\left(1 - \frac{y}{3}\right) - 2xy.$$

Nullclines

We find y-nullclines. (Detail 2)

Nullclines

We find y-nullclines. (Detail 2)

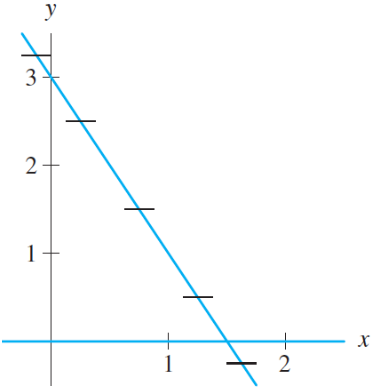

Figure 5.14
The y-nullclines for the system

$$\frac{dx}{dt} = 2x\left(1 - \frac{x}{2}\right) - xy$$
$$\frac{dy}{dt} = 3y\left(1 - \frac{y}{3}\right) - 2xy.$$

Nullclines

Putting together, we get (Detail 3)

Nullclines

Putting together, we get (Detail 3)

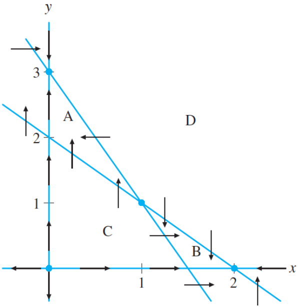

Figure 5.15
The x- and y-nullclines for

$$\frac{dx}{dt} = 2x\left(1 - \frac{x}{2}\right) - xy$$
$$\frac{dy}{dt} = 3y\left(1 - \frac{y}{3}\right) - 2xy.$$

The nullclines divide the first quadrant into four regions marked A, B, C, and D.

Nullclines

Putting together, we get (Detail 3)

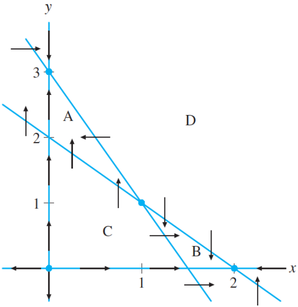

Figure 5.15
The x- and y-nullclines for

$$\frac{dx}{dt} = 2x\left(1 - \frac{x}{2}\right) - xy$$
$$\frac{dy}{dt} = 3y\left(1 - \frac{y}{3}\right) - 2xy.$$

The nullclines divide the first quadrant into four regions marked A, B, C, and D.

Conclude that solutions that do not tend to $(1,1)$ tend either to $(0,3)$ or to $(2,0)$.

Nullclines

Putting together, we get (Detail 3)

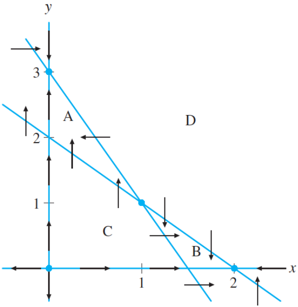

Figure 5.15
The x- and y-nullclines for

$$\frac{dx}{dt} = 2x\left(1 - \frac{x}{2}\right) - xy$$
$$\frac{dy}{dt} = 3y\left(1 - \frac{y}{3}\right) - 2xy.$$

The nullclines divide the first quadrant into four regions marked A, B, C, and D.

Conclude that solutions that do not tend to $(1, 1)$ tend either to $(0, 3)$ or to $(2, 0)$.
Warning) Nullclines are not necessarily solutions.

Exercise

5 Sketch the phase portrait (in the first quadrant) of

$$\frac{dx}{dt} = x(-x - 3y + 150)$$
$$\frac{dy}{dt} = y(-2x - y + 100)$$

based on nullclines and linearization.

Nullclines That Are Not Lines

Consider

$$\frac{dx}{dt} = 2x\left(1 - \frac{x}{2}\right) - xy$$
$$\frac{dy}{dt} = y\left(\frac{9}{4} - y^2\right) - x^2y.$$

([PRG], p.482)

Nullclines That Are Not Lines

We can use nullclines to get some idea bout the direction field. (Detail 4)

Nullclines That Are Not Lines

We can use nullclines to get some idea bout the direction field. (Detail 4)

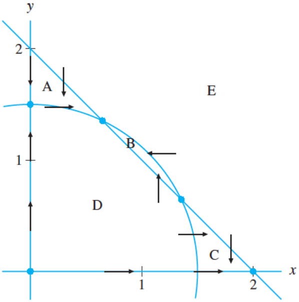

Figure 5.17
Nullclines for the system

$$\frac{dx}{dt} = 2x\left(1 - \frac{x}{2}\right) - xy$$
$$\frac{dy}{dt} = y\left(\frac{9}{4} - y^2\right) - x^2 y.$$

The nullclines separate the first quadrant into five regions.

Nullclines That Are Not Lines

We can use nullclines to get some idea bout the direction field. (Detail 4)

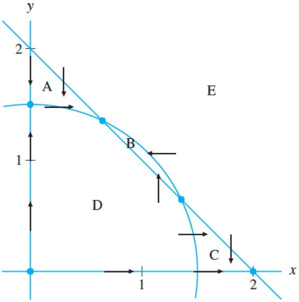

Figure 5.17
Nullclines for the system

$$\frac{dx}{dt} = 2x\left(1 - \frac{x}{2}\right) - xy$$
$$\frac{dy}{dt} = y\left(\frac{9}{4} - y^2\right) - x^2 y.$$

The nullclines separate the first quadrant into five regions.

We can draw a rough sketch of the phase portrait. (Detail 5)
([PRG], p.482)

Exercise

Review Exercise for Chapter 5 on page 556

15

Sketch the phase portrait of the system

$$\frac{dx}{dt} = x - 3y^2$$
$$\frac{dy}{dt} = x - 3y - 6.$$

Using All Our Tools

Consider

$$\frac{dx}{dt} = x + y - x^3$$
$$\frac{dy}{dt} = -0.5x.$$

- Information from linearization (Detail 6) is

Using All Our Tools

Consider

$$\frac{dx}{dt} = x + y - x^3$$
$$\frac{dy}{dt} = -0.5x.$$

- Information from linearization (Detail 6) is that the origin is a spiral source.
- Information from nullclines is...

([PRG], p.485)

Using All Our Tools

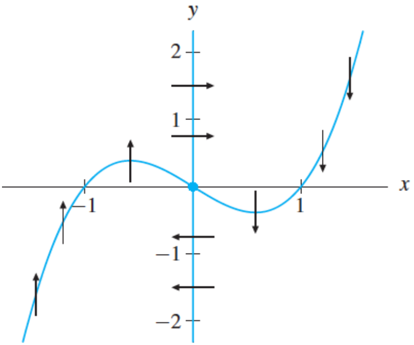

Figure 5.19
The x- and y-nullclines for the system

$$\frac{dx}{dt} = x + y - x^3$$
$$\frac{dy}{dt} = -0.5x.$$

The nullclines for this system separate the plane into four regions.

Therefore, we want to draw (Detail 7). However numerics show us...

Using All Our Tools

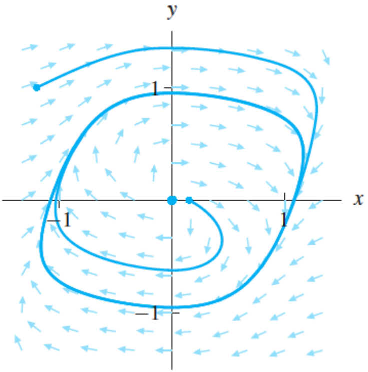

Figure 5.20
Solutions of the system

$$\frac{dx}{dt} = x + y - x^3$$
$$\frac{dy}{dt} = -0.5x.$$

Note that solutions with initial conditions close to the origin spiral outward, while those with initial conditions far from the origin spiral inward.

Using All Our Tools

- Based on *linearization*, the origin is locally a spiral source.

Using All Our Tools

- Based on *linearization*, the origin is locally a spiral source.
- Based on *nullcline*, all solutions must circulate clockwise around the origin

Using All Our Tools

- Based on *linearization*, the origin is locally a spiral source.
- Based on *nullcline*, all solutions must circulate clockwise around the origin
- Based on *numerics*, far from the origin, solutions spiral inward.

Using All Our Tools

- Based on *linearization*, the origin is locally a spiral source.
- Based on *nullcline*, all solutions must circulate clockwise around the origin
- Based on *numerics*, far from the origin, solutions spiral inward.
- From the Uniqueness Theorem, solution curves never cross.

Using All Our Tools

- Based on *linearization*, the origin is locally a spiral source.
- Based on *nullcline*, all solutions must circulate clockwise around the origin
- Based on *numerics*, far from the origin, solutions spiral inward.
- From the Uniqueness Theorem, solution curves never cross.
- Therefore, there must be at least one periodic solution that spirals neither outward nor inward.

Using All Our Tools

- Based on *linearization*, the origin is locally a spiral source.
- Based on *nullcline*, all solutions must circulate clockwise around the origin
- Based on *numerics*, far from the origin, solutions spiral inward.
- From the Uniqueness Theorem, solution curves never cross.
- Therefore, there must be at least one periodic solution that spirals neither outward nor inward.

Exercise

Review Exercise for Chapter 5 on page 557
26
Consider

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} - 3x + x^3 = 0.$$

Sketch the phase portrait.

Overview

1. Sect. 5.2 Qualitative Analysis
   - Competing Species
   - Nullclines
   - Nullclines That Are Not Lines
   - Using All Our Tools
   - Homework

What's next: Sect. 5.3 Hamiltonian Systems

Homework

- Suggested Exercises (optional): 1, 5 , 21, 22, 23
- Homework Exercises (required to submit): 1, 5 except (c), 21, 23

References

Paul Blanchard, Robert L. Devaney, Glen R. Hall
Differential Equations, fourth edition.