Chapter 5 Nonlinear Systems

Sect. 5.2 Qualitative Analysis

Jeaheang(Jay) Bang

Rutgers University

j.bang@rutgers.edu

Thu. Aug 10, 2017

Overview

- Question 1) How can we draw the phase portrait of (Detail 1)

$$1)\frac{dx}{dt} = y \qquad\qquad 2)\frac{dy}{dt} = v$$
$$\frac{dy}{dt} = x - x^2 \qquad\qquad \frac{dv}{dt} = -2x$$

- Question 2) Did you have a dream as a kid? My dream was

my childish dream

Overview

Hamiltonian Systems

Conserved Quantity

- A real-valued function $H(x, y)$ of the two variables x and y is a **conserved quantity** for a system of differential equations if it is constant along all solution curves of the system.
- That is if $(x(t), y(t))$ is a solution of the system, then $H(x(t), y(t))$ is constant. In other words,

$$\frac{d}{dt} H(x(t), y(t)) = 0.$$

([PRG], p.494)

Hamiltonian Systems

- Consider

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = x - x^2$$

- Define

$$H(x, y) = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{3}x^3.$$

- For solutions (x, y),

$$\frac{d}{dt}H(x(t), y(t)) = 0. \quad \text{(Detail 2)}$$

- The solution curves always lie along the level curves of H.

Hamiltonian Systems

The Level curves of H.

The phase portrait

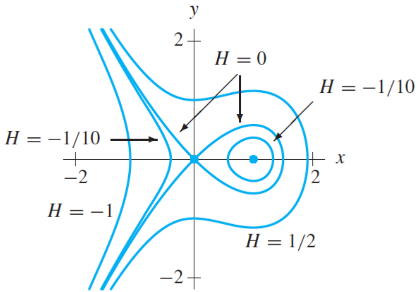


Figure 5.25

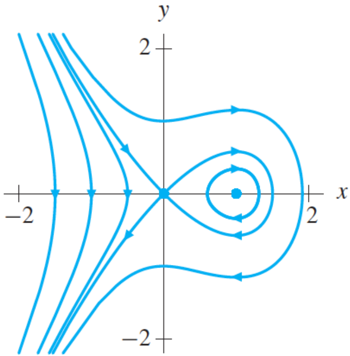


Figure 5.24

Now we can draw phase portrait. (Detail 3)

Hamiltonian Systems

Definition

- A system of differential equations is called a **Hamiltonian system** if there exists a real-valued function $H(x, y)$ such that

$$\frac{dx}{dt} = \frac{\partial H}{\partial y}$$
$$\frac{dy}{dt} = -\frac{\partial H}{\partial x}$$

  for all x, y.
- The function H is called the **Hamiltonian function** for the system.

Hamiltonian System

- Note H is always a conserved quantity for such a system. (Detail 4)
- Sketching the phase portrait for a Hamiltonian system is the same as sketching the level sets of the Hamiltonian function.

Examples: The Harmonic Oscillator

- Recall that the undamped harmonic oscillator system is

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = -qy$$

  where q is a positive constant.
- Let

$$H(y, v) = \frac{1}{2}v^2 + \frac{q}{2}y^2.$$

- Then

$$\frac{dy}{dt} = \frac{\partial H}{\partial v}, \quad \frac{dv}{dt} = -\frac{\partial H}{\partial y}.$$

- Hence the undamped harmonic oscillator is a Hamiltonian system.

([PRG], p.494)

Examples: The Harmonic Oscillator

- Going back to the first question

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = -2x,$$

  a Hamiltonian function is

$$H(y, v) = \frac{1}{2}v^2 + y^2.$$

- Therefore, the solution curves are ellipses.

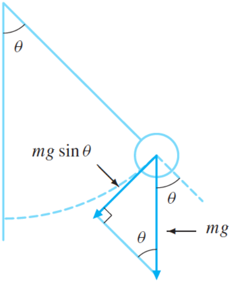
The Nonlinear Pendulum

Consider a pendulum.

Figure 5.27
A pendulum with rod length l and angle θ.

([PRG], p.495)

We assume there are only two forces: gravity and friction.



The Nonlinear Pendulum

- Using Newton's second law,

$$-bl\frac{d\theta}{dt} - mg\sin\theta = ml\frac{d^2\theta}{dt^2}$$

  where b is the coefficient of damping,
- which is often written as

$$\frac{d^2\theta}{dt^2} + \frac{b}{m}\frac{d\theta}{dt} + \frac{g}{l}\sin\theta = 0.$$

- Suppose $b = 0$ and $l = 1$.

The Nonlinear Pendulum

We rewrite the equation as a first-order system in the usual manner to get

$$\begin{aligned}\frac{d\theta}{dt} &= v \\ \frac{dv}{dt} &= -g\sin\theta.\end{aligned}$$

- Find equilibrium points: $(n\pi, 0)$ where n is any integer.
- Let
$$H(\theta, v) = \frac{1}{2}v^2 - g\cos\theta.$$
-

The Nonlinear Pendulum

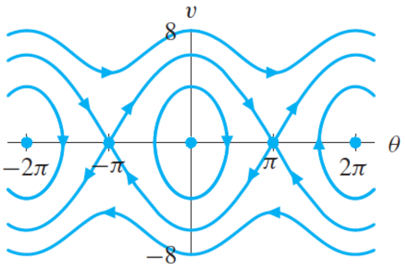


Figure 5.30
Phase portrait for the ideal pendulum.

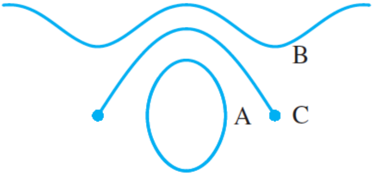


Figure 5.31
Special solution curves.

▸ my childish dream

Finding Hamiltonian Systems

Study how to determine if a given system is a Hamiltonian system.

- Consider

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y).$$

- If the system is Hamiltonian,

$$\frac{\partial f}{\partial x} = -\frac{\partial g}{\partial y} \quad \text{(Detail 5)}$$

- The converse is also true.

([PRG], p.498)

Finding Hamiltonian Systems

- Examples (Detail 6)

$$1)\frac{dx}{dt} = x + y^2 \quad 2)\frac{dx}{dt} = y \quad 3)\frac{dx}{dt} = -x\sin y + 2y$$
$$\frac{dy}{dt} = y^2 - x \quad \frac{dy}{dt} = x - x^2 \quad \frac{dy}{dt} = -\cos y.$$

- \# 14
  Show the system

$$\frac{dx}{dt} = F(y)$$
$$\frac{dy}{dt} = G(x).$$

is Hamiltonian.

Finding Hamiltonian Systems

Review Exercise for Chapter 5
28
Consider the linear system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathbf{Y}.$$

For which values of a, b, c, and d is the system Hamiltonian?

Equilibrium Points of Hamiltonian Systems

Hamiltonian systems have a number of special properties not shared by general systems.
Question) Can you tell which one is Hamiltonian?

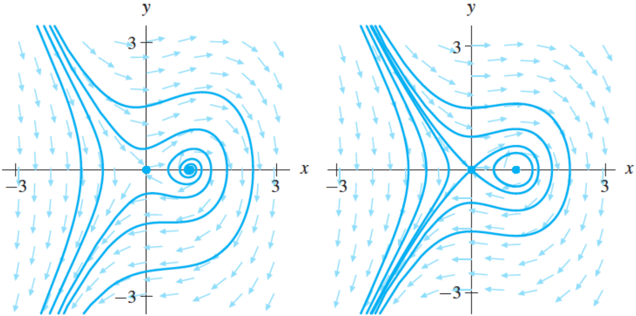


Figure 5.33

Equilibrium Points of Hamiltonian Systems

- For a Hamiltonian system, the Jacobian matrix assumes the form

$$\begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}. \quad \text{(Detail 7)}$$

- Thus there are only three possibilities: (Detail 8)
  1. The equilibrium point is a saddle
  2. The equilibrium point is a center.
  3. The only eigenvalue is 0.

([PRG], p.502)

Equilibrium Points of Hamiltonian Systems

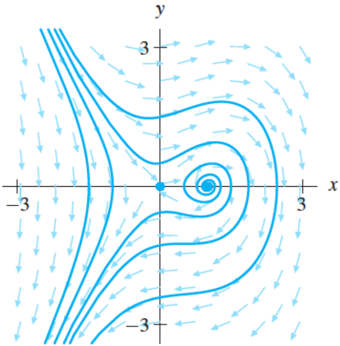


Figure 5.33
The phase portrait on the left cannot be a Hamiltonian system, whereas the phase portrait on the right might be Hamiltonian.

Overview

What's next: Sect. 5.4 Dissipative Systems

Homework

- Suggested Exercises (optional): 1, 9, 11, 13, 14, 15, 17
- Homework Exercises (required to submit): 1, 9, 11, 17

References

Paul Blanchard, Robert L. Devaney, Glen R. Hall
Differential Equations, fourth edition.