Chapter 5 Nonlinear Systems

Sect. 5.4 Dissipative Systems

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Overview

Recall

- a pendulum is described by

$$\frac{d\theta}{dt} = v$$
$$\frac{dv}{dt} = -\frac{b}{m}v - g\sin\theta.$$

- If $b = 0$ (no damping), it is Hamiltonian with

$$H(\theta, v) = \frac{1}{2}v^2 - g\cos\theta.$$

- But what if $b \neq 0$? It is no longer Hamiltonian.

Overview

Recall that

- the damped harmonic oscillator system is

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = -qy - pv.$$

- If $p = 0$ (no damping), then it is Hamiltonian with

$$H(y, v) = \frac{1}{2}v^2 + \frac{q}{2}y^2.$$

- But what if $p \neq 0$? It is no longer Hamiltonian.

Overview

The Nonlinear Pendulum with Friction

Study

$$\frac{d\theta}{dt} = v$$
$$\frac{dv}{dt} = -\frac{b}{m}v - \frac{g}{l}\sin\theta$$

for small b.

- Equilibrium Points are $(n\pi, 0)$ for any integer n.
- Linearization (Detail 1) yields $(2n\pi, 0)$ are spiral sinks and $((2n+1)\pi, 0)$ are saddles.
- We can also consider nullclines. (Detail 2)

([PRG], p.508)

The Nonlinear Pendulum with Friction

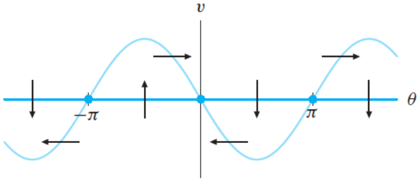

Figure 5.34
Nullclines for the system

$$\frac{d\theta}{dt} = v$$
$$\frac{dv}{dt} = -\frac{b}{m}v - \frac{g}{l}\sin\theta.$$

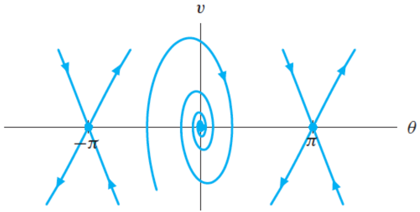

Figure 5.35
Phase portrait near the equilibrium points for the system

$$\frac{d\theta}{dt} = v$$
$$\frac{dv}{dt} = -\frac{b}{m}v - \frac{g}{l}\sin\theta.$$

The Effects of Dissipation

- We do not have the benefit of a conserved quantity as we did in the frictionless case.
- We still use the Hamiltonian function that we developed for the ideal pendulum.

$$H(\theta, v) = \frac{1}{2}v^2 - \frac{g}{l}\cos\theta.$$

- Note
$$\frac{dH}{dt} \leq 0. \qquad \text{(Detail 3)}$$
- Hence the solution curves in the θv-plane cross level sets of H moving from larger to smaller H values. (Detail 4)

([PRG], p.512)

The Effects of Dissipation

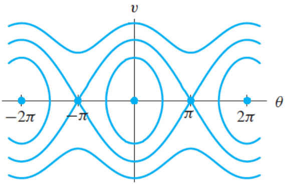

Figure 5.36
Level curves of

$$H = \frac{1}{2}v^2 - \frac{g}{l}\cos\theta.$$

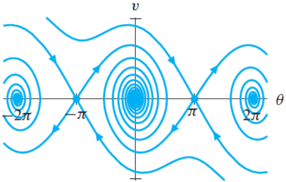

Figure 5.37
Phase portrait for the system

$$\frac{d\theta}{dt} = v$$
$$\frac{dv}{dt} = \frac{b}{m}v - \frac{g}{l}\sin\theta.$$

Lyapunov Functions

Definition

A function $L(x, y)$ is called a **Lyapunov function** for a system of differential equations if for every non-equilibrium solution $(x(t), y(t))$,

$$\frac{d}{dt}L(x(t), y(t)) \leq 0$$

for all t with strict inequality except for a discrete set of t's.

([PRG], p.513)

The Damped Harmonic Oscillator

Consider

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = -qy - pv.$$

Claim it possesses a Lyapunov function.

- Recall a Hamiltonian function for the case $p = 0$ is $H(y, v) = \frac{1}{2}v^2 + \frac{q}{2}y^2$.
- For the general case $p > 0$,
$$\frac{d}{dt}H(y(t), v(t)) \leq 0,$$
- This implies H is a Lyapunov function for the damped harmonic oscillator.

([PRG], p.513)

The Damped Harmonic Oscillator

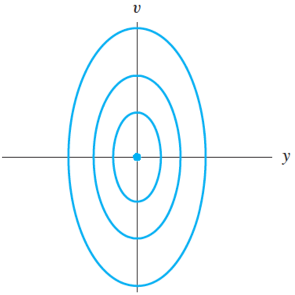

Figure 5.39
Level sets for $H(y, v) = \frac{1}{2}(v^2 + qy^2)$. With the exception of the point at the origin, each level set is an ellipse.

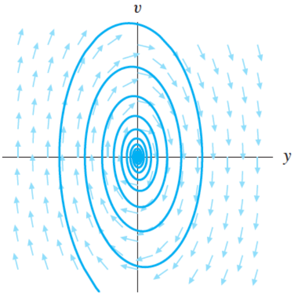

Figure 5.40
Phase portrait for the damped harmonic oscillator with p small.

Exercise

1
Consider

$$\frac{dx}{dt} = -x^3$$
$$\frac{dy}{dt} = -y^3.$$

Question) Can we find a Lyapunov function? Do we have some systematic way to find a Lyapunov function?
Verify that $L(x, y) = (x^2 + y^2)/2$ is a Lyapunov function for the system.

The Tuned-Mass Damper

▸ Tuned-Mass Damper ▸ Taipei 101

We consider a simple model.

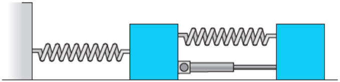

Figure 5.41
A mass-spring system with a tuned-mass damper.

Quantities

- x_1, x_2: the horizontal positions of the springs measured from the wall.
- m_1, m_2: the masses, k_1, k_2: the spring constants.
- L_1, L_2 : the rest lengths of the two springs.

([PRG], p.515)

The Tuned-Mass Damper

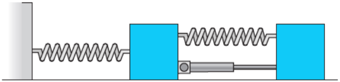

Figure 5.41
A mass-spring system with a tuned-mass damper.

We first model the system without damping. We obtain the equations

$$m_1 \frac{d^2x_1}{dt^2} = -k_1(x_1 - L_1) + k_2(x_2 - x_1 - L2)$$
$$m_2 \frac{d^2x_2}{dt^2} = -k_2(x_2 - x_1 - L_2)$$

We turn this into a system...

The Tuned-Mass Damper

The equations turn into

$$\frac{dx_1}{dt} = \frac{p_1}{m_1}$$
$$\frac{dx_2}{dt} = \frac{p_2}{m_2}$$
$$\frac{dp_1}{dt} = -k_1(x_1 - L_1) + k_2(x_2 - x_1 - L_2)$$
$$\frac{dp_2}{dt} = -k_2(x_2 - x_1 - L_2).$$

Then it is Hamiltonian with the Hamiltonian function

$$H(x_1, x_2, p_1, p_2) = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{k_1}{2}(x_1 - L_1)^2 + \frac{k_2}{2}(x_2 - x_1 - L_2)^2$$

The Tuned-Mass Damper

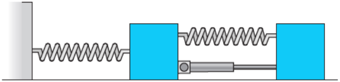

Figure 5.41
A mass-spring system with a tuned-mass damper.

Now we consider the damping. Then the equations become

$$m_1 \frac{d^2 x_1}{dt^2} = -k_1(x_1 - L_1) + k_2(x_2 - x_1 - L_2) + b\left(\frac{dx_2}{dt} - \frac{dx_1}{dt}\right)$$
$$m_2 \frac{d^2 x_2}{dt^2} = -k_2(x_2 - x_1 - L_2) - b\left(\frac{dx_2}{dt} - \frac{dx_1}{dt}\right).$$

Then the system is no longer Hamiltonian. Compute

$$\frac{dH}{dt} = -b\left(\frac{dx_2}{dt} - \frac{dx_1}{dt}\right)^2 \leq 0, \qquad \text{(Detail 5)}$$

Therefore, we conclude that

> *the energy decreases whenever the second mass is moving relative to the first.*

Gradient Systems

Consider

$$\frac{dx}{dt} = x - x^3$$
$$\frac{dy}{dt} = -y.$$

Question) does it possess a Lyapunov function? If yes, how can find one? ([PRG], p.519)

Gradient Systems

The system has a special structure;

$$\frac{dx}{dt} = x - x^3 = \frac{\partial S}{\partial x}$$
$$\frac{dy}{dt} = -y = \frac{\partial S}{\partial y}$$

where S is given by

$$S(x, y) = \frac{x^2}{2} - \frac{x^4}{4} - \frac{y^2}{2}.$$

Compute

$$\frac{d}{dt} S(x(t), y(t)) \geq 0, \qquad \text{(Detail 6)}$$

Now we can get a fairly complete phase portrait. (Detail 7)

Gradient Systems

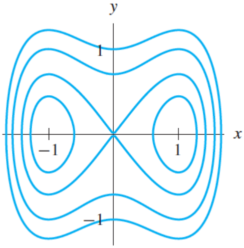

Figure 5.44
Level sets of

$$S(x, y) = \frac{x^2}{2} - \frac{x^4}{4} - \frac{y^2}{2} + 8.$$

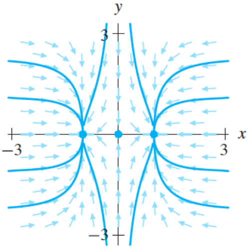

Figure 5.45
Phase portrait for the system

$$\frac{dx}{dt} = \frac{\partial S}{\partial x} = x - x^3$$
$$\frac{dy}{dt} = \frac{\partial S}{\partial y} = -y.$$

General Form of Gradient Systems

Definition

A system of differential equations is called a **gradient system** if there is a function G such that

$$\frac{dx}{dt} = \frac{\partial G}{\partial x}$$
$$\frac{dy}{dt} = \frac{\partial G}{\partial y}$$

for all (x, y).

Then $-G$ is a Lyapunov function. (Detail 8) ([PRG], p.522)

Exercise

13

Sketch the phase portrait of

$$\frac{dx}{dt} = 2x$$
$$\frac{dy}{dt} = -2y.$$

Properties of Gradient Systems

There are restrictions on the types of equilibrium points that occur in a gradient system.

- For a gradient system, the Jacobian matrix assumes the form

$$\begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \qquad \text{(Detail 9)}$$

- Therefore, gradient systems do not have spiral sinks, spiral sources, or centers. (Detail 10)

([PRG], p.523)

Properties of Gradient Systems

Not all systems of differential equations that possess Lyapunov functions are gradient systems.

- Consider

$$\frac{dx}{dt} = -x + y$$
$$\frac{dy}{dt} = -x - y.$$

- A Lyapunov function is given by $L(x, y) = x^2 + y^2$. (Detail 11)
- But the system cannot be a gradient system.

Overview

1. Sect. 5.4 Dissipative Systems
   - The Nonlinear Pendulum with Friction
   - The Effects of Dissipation
   - Lyapunov Functions
   - The Tuned-Mass Damper
   - Gradient Systems
   - Properties of Gradient Systems

What's next: Review Session and Final Exam

Homework

- Suggested Exercises (optional): 1, 3, 13, 15, 19

References

Paul Blanchard, Robert L. Devaney, Glen R. Hall
Differential Equations, fourth edition.